From Determinism to Stochasticity
Measurement Theory II

Reading for this lecture:

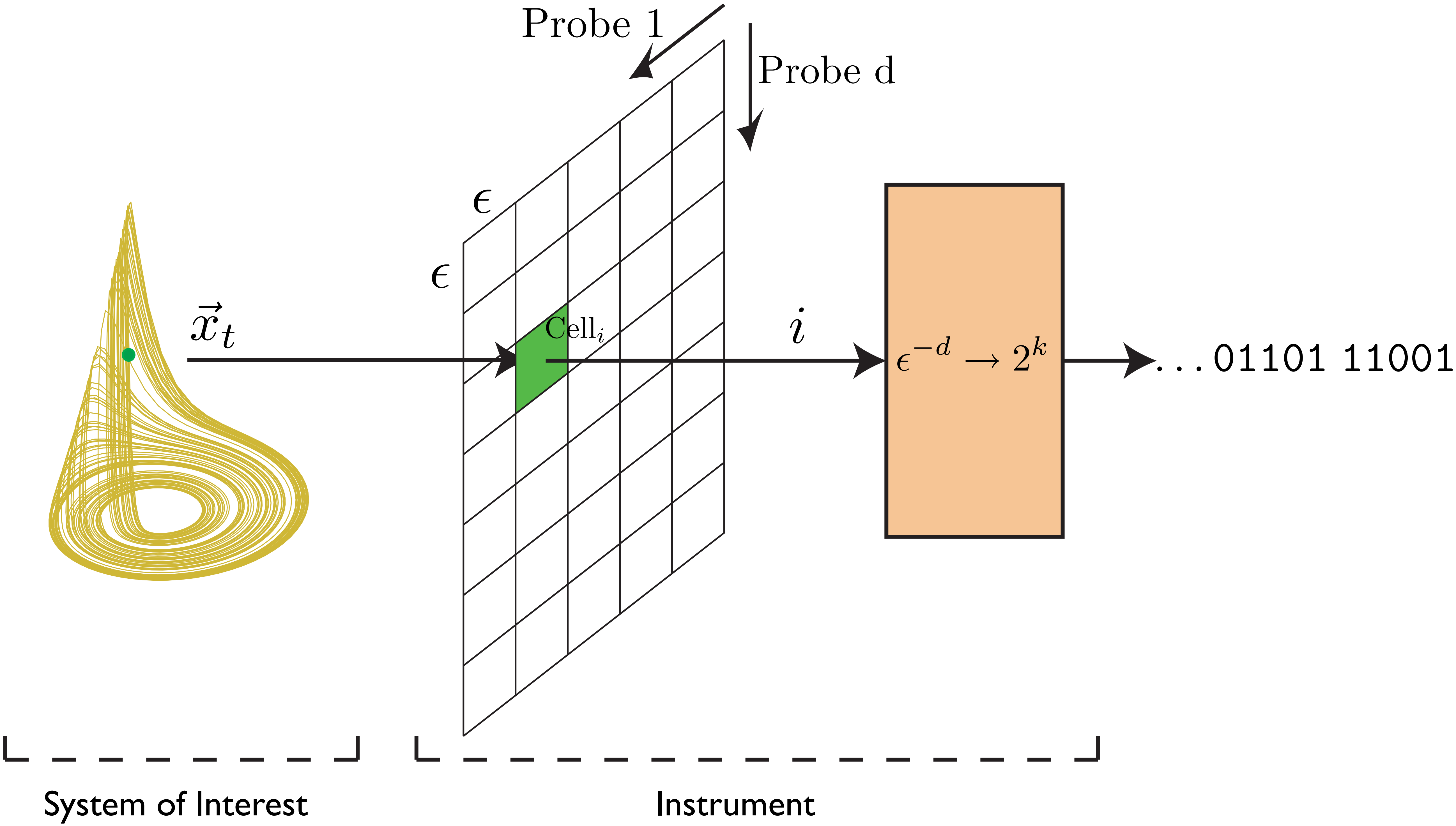
(These) *Lecture Notes*.

Interactive Labs:

Symbolic Dynamics: From Continuous to Discrete

Symbolic Dynamics: Symbolic Construction Examples

From Determinism to Stochasticity ...



Measurement Channel

From Determinism to Stochasticity ...

Measurement Theory ...

Markov Partitions for higher dimensional systems:

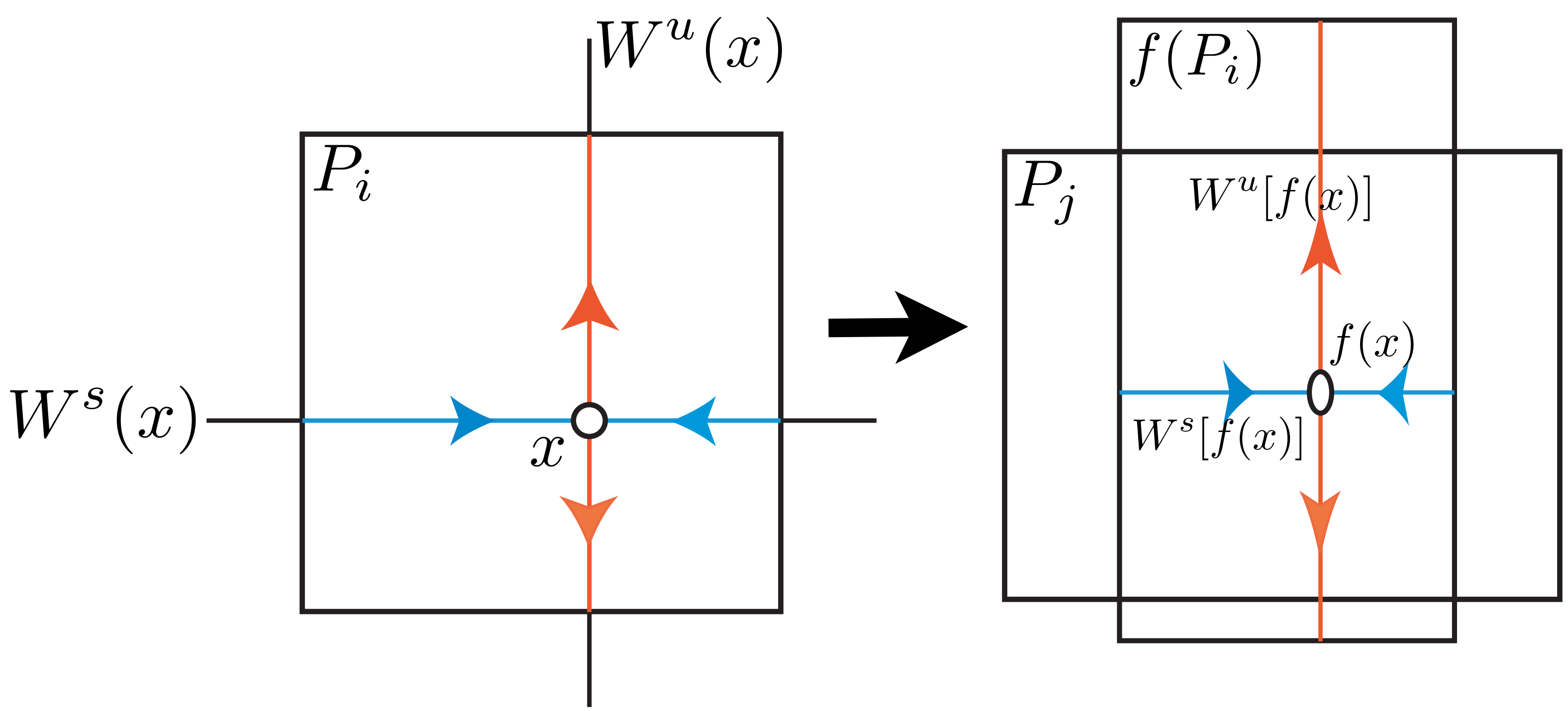
Partition $\mathcal{P}$ is a Markov when

$$x \in P_i \to f(x) \in P_j$$

then $\quad f[W^u(x) \cap P_i] \supset W^u[f(x)] \cap P_j$

$\forall i$

and $\quad f[W^s(x) \cap P_i] \subset W^s[f(x)] \cap P_j$

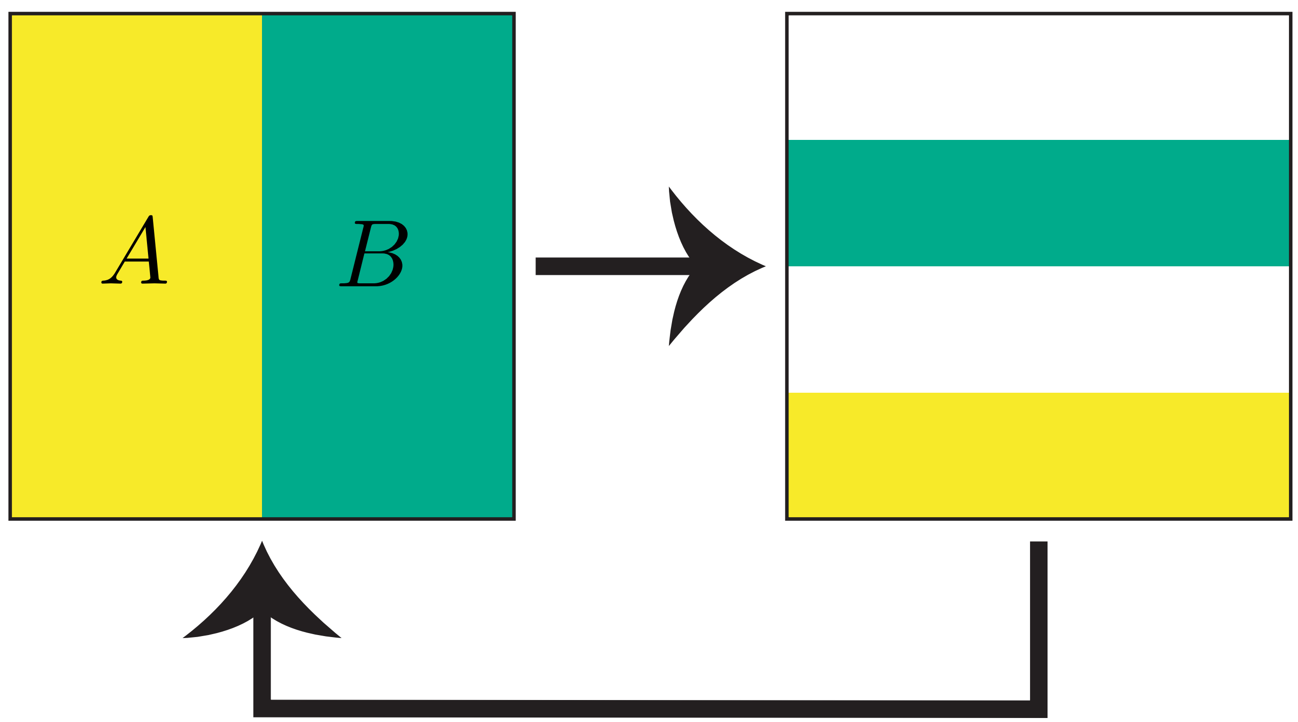


From Determinism to Stochasticity ...

Measurement Theory ...

Markov partition for 2D Dissipative Baker's Map:

$$
\begin{aligned}
x_{n+1} &= 2x_n \pmod 1 \\
y_{n+1} &= \begin{cases} \frac{a}{2} y_n, & x_n \leq \frac{1}{2} \\ \frac{1}{2} + \frac{a}{2} y_n, & x_n > \frac{1}{2} \end{cases}
\end{aligned}
$$



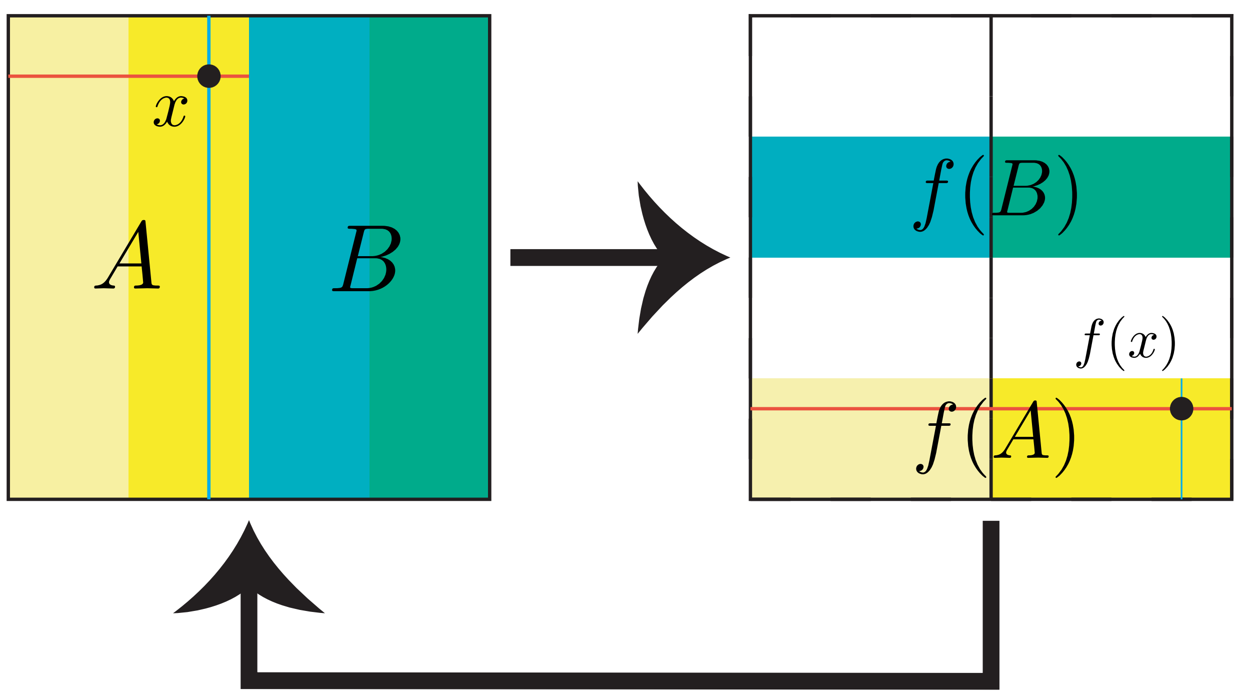
$$\mathcal{P} = \{A \sim [0, \tfrac{1}{2}] \times [0,1], B \sim [\tfrac{1}{2}, 1] \times [0,1]\}$$

From Determinism to Stochasticity ...

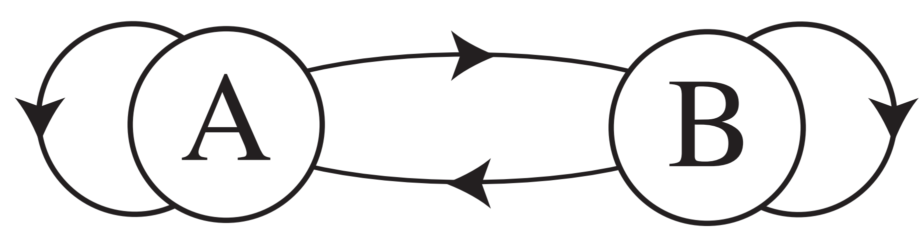
Measurement Theory ...

Markov partition for 2D Dissipative Baker's Map:

$$\mathcal{P} = \{A \sim [0, \tfrac{1}{2}] \times [0, 1], B \sim [\tfrac{1}{2}, 1] \times [0, 1]\}$$



Markov chain:

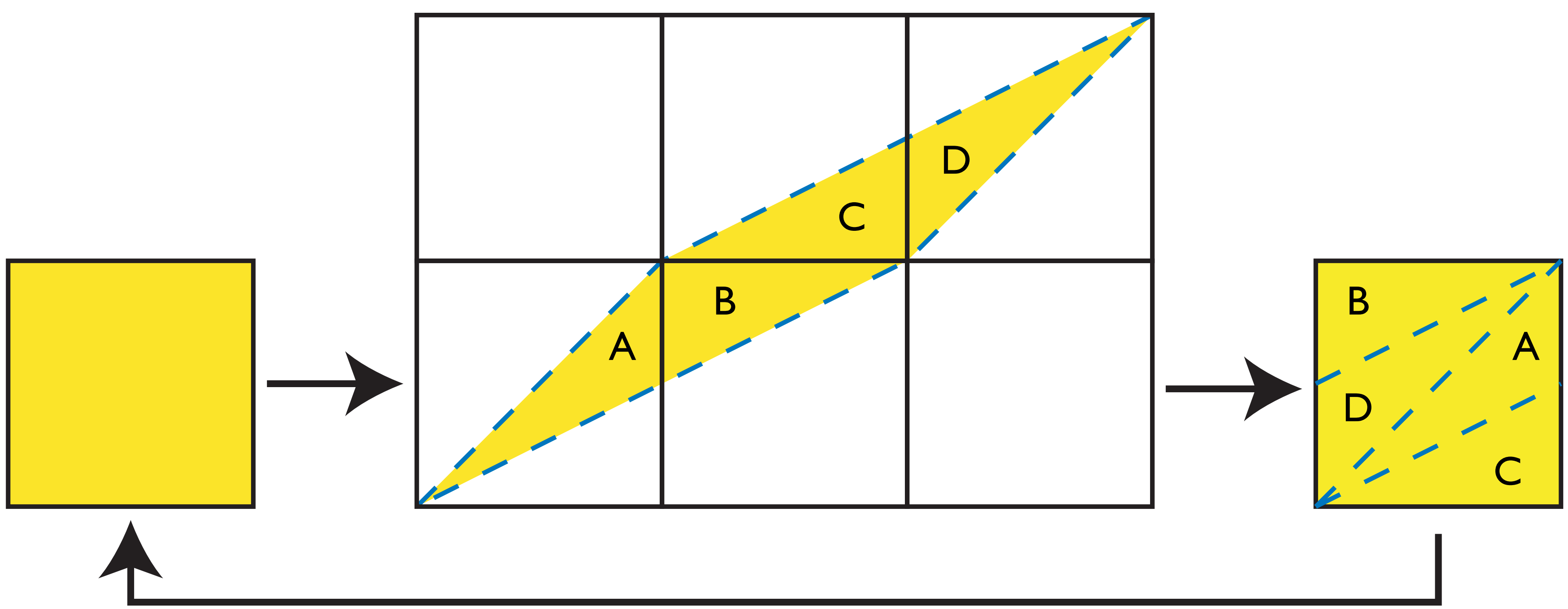


From Determinism to Stochasticity ...

Measurement Theory ...

Markov partition for 2D Cat Map ...

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} \pmod 1$$



From Determinism to Stochasticity ...

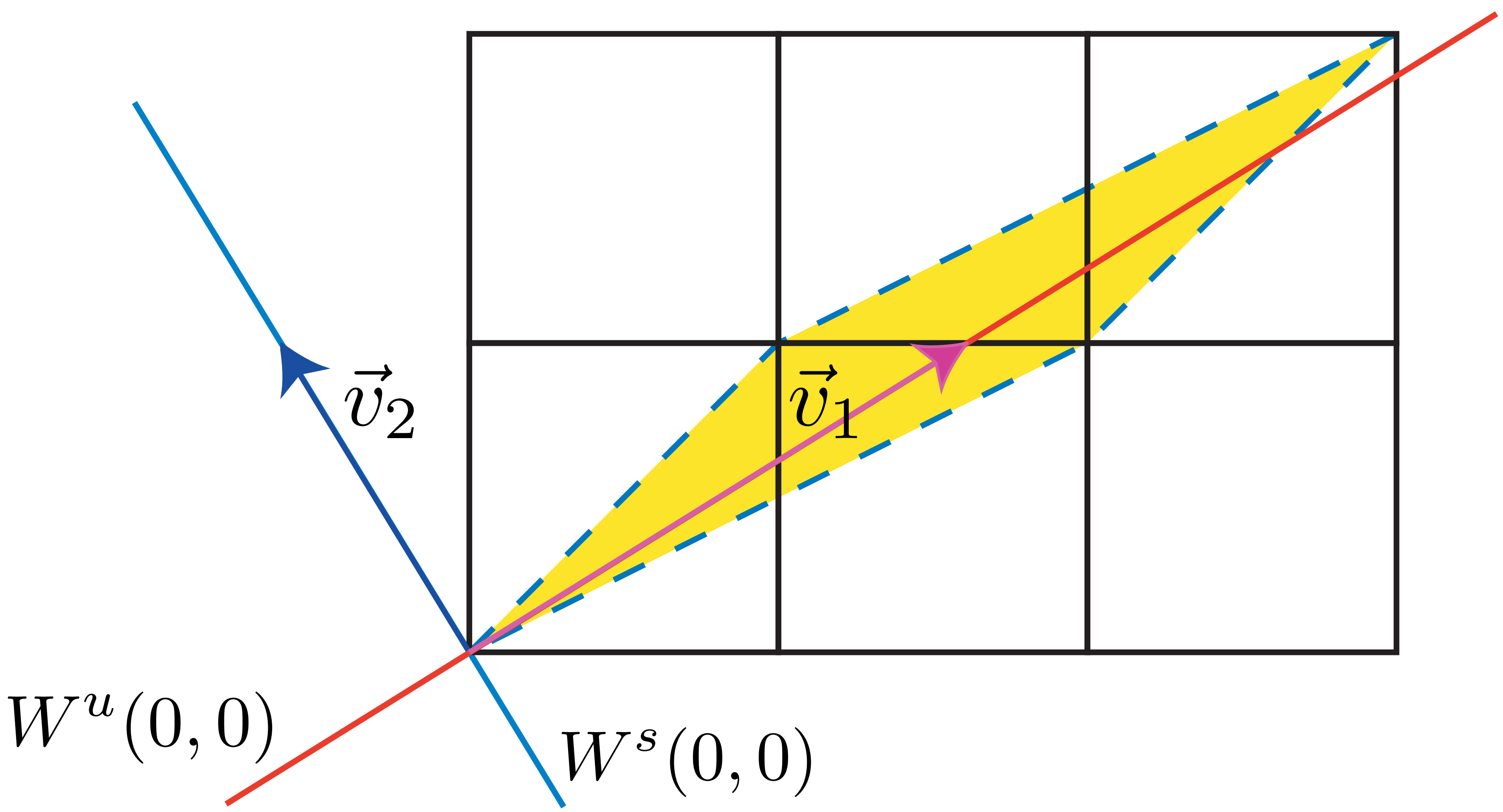
Measurement Theory ...

Markov partition for 2D Cat Map:

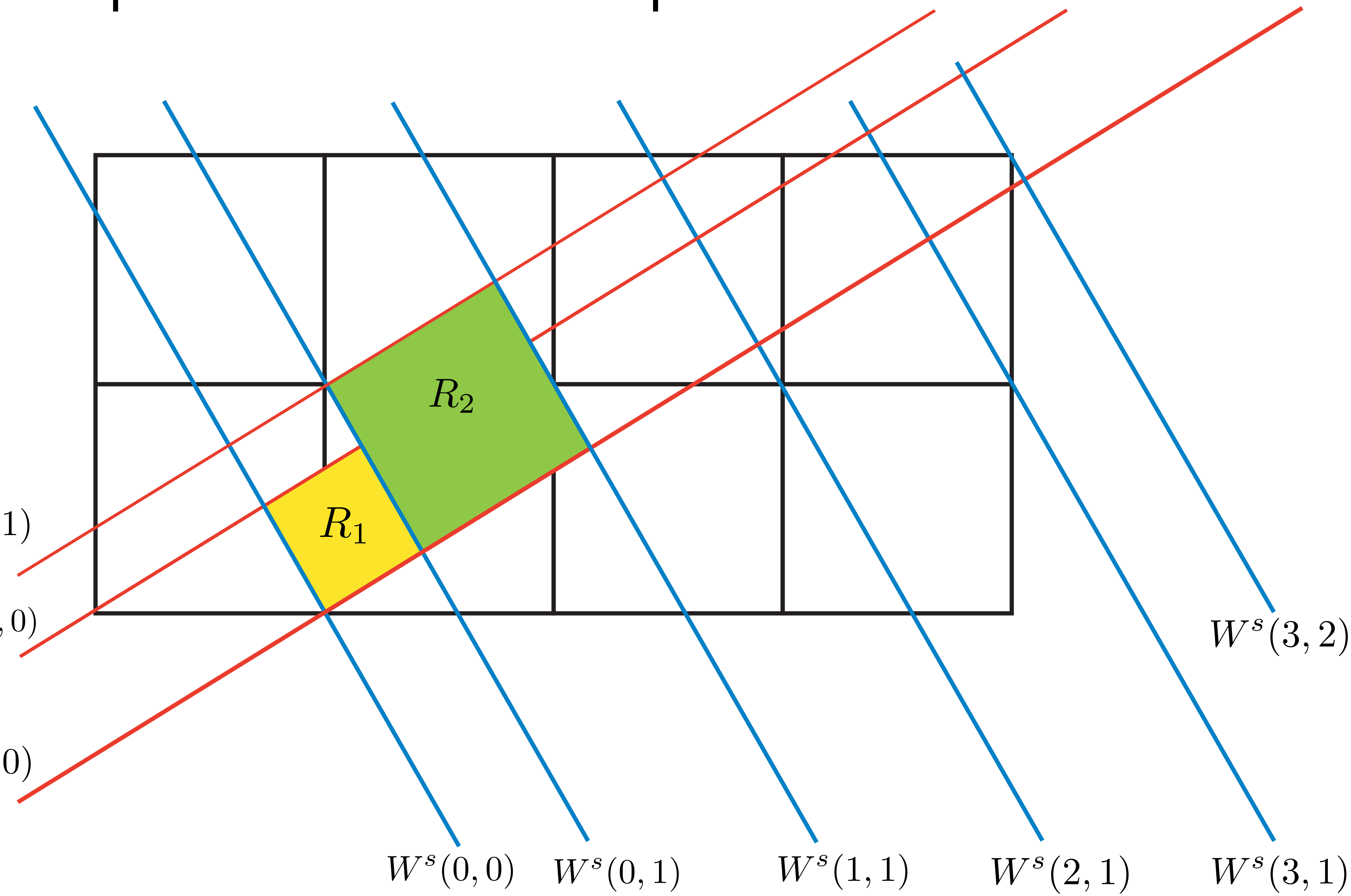
Fixed point: $\vec{x}^* = (0, 0)$

Eigensystem: $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ stretch $\quad \vec{v}_1 = (\frac{1+\sqrt{5}}{2}, 1)$

$\lambda_2 = \frac{3-\sqrt{5}}{2} < 1$ shrink $\quad \vec{v}_2 = (\frac{1-\sqrt{5}}{2}, 1)$



From Determinism to Stochasticity ...
Measurement Theory ...
Markov partition for 2D Cat Map ...

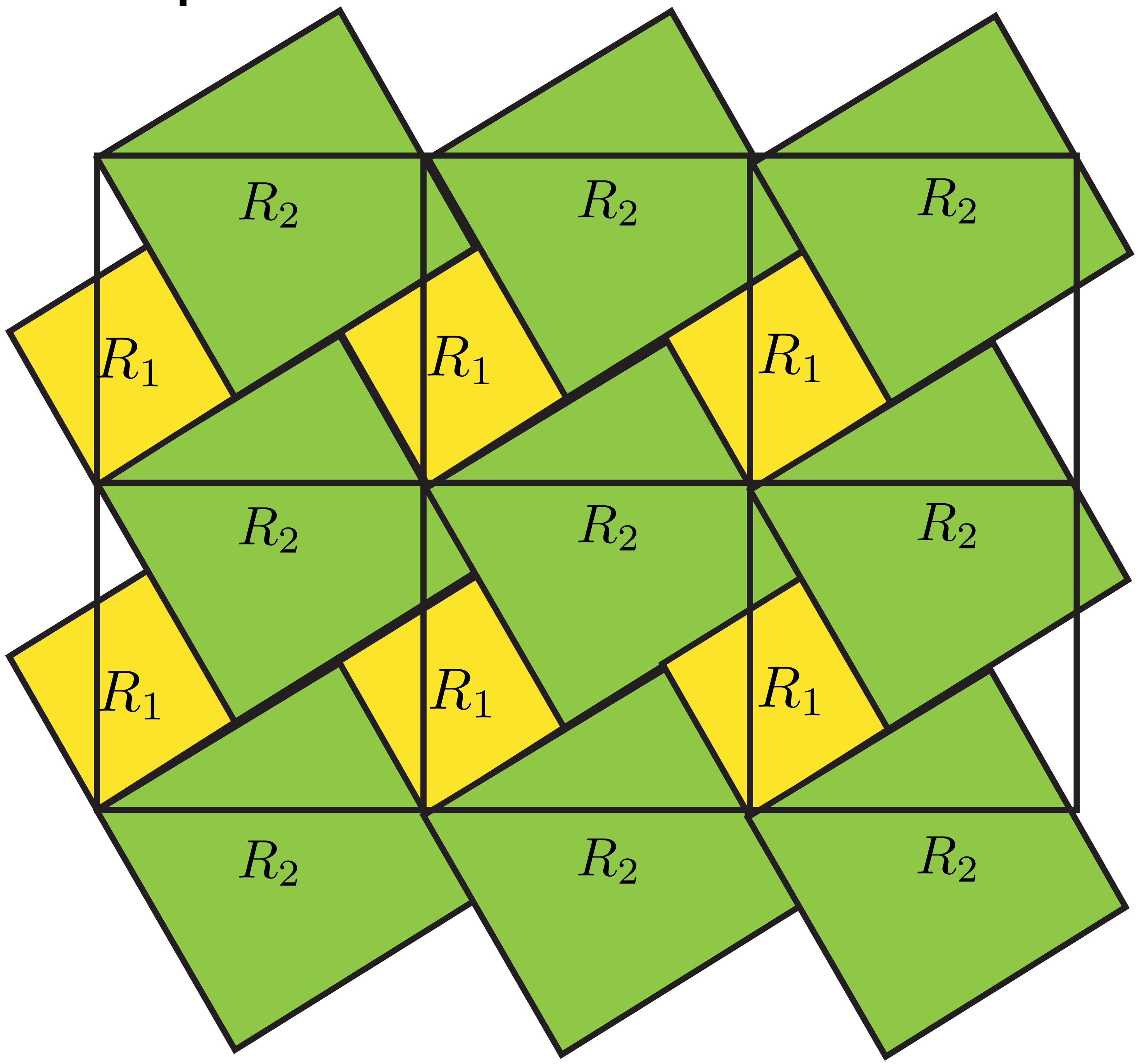


From Determinism to Stochasticity ...

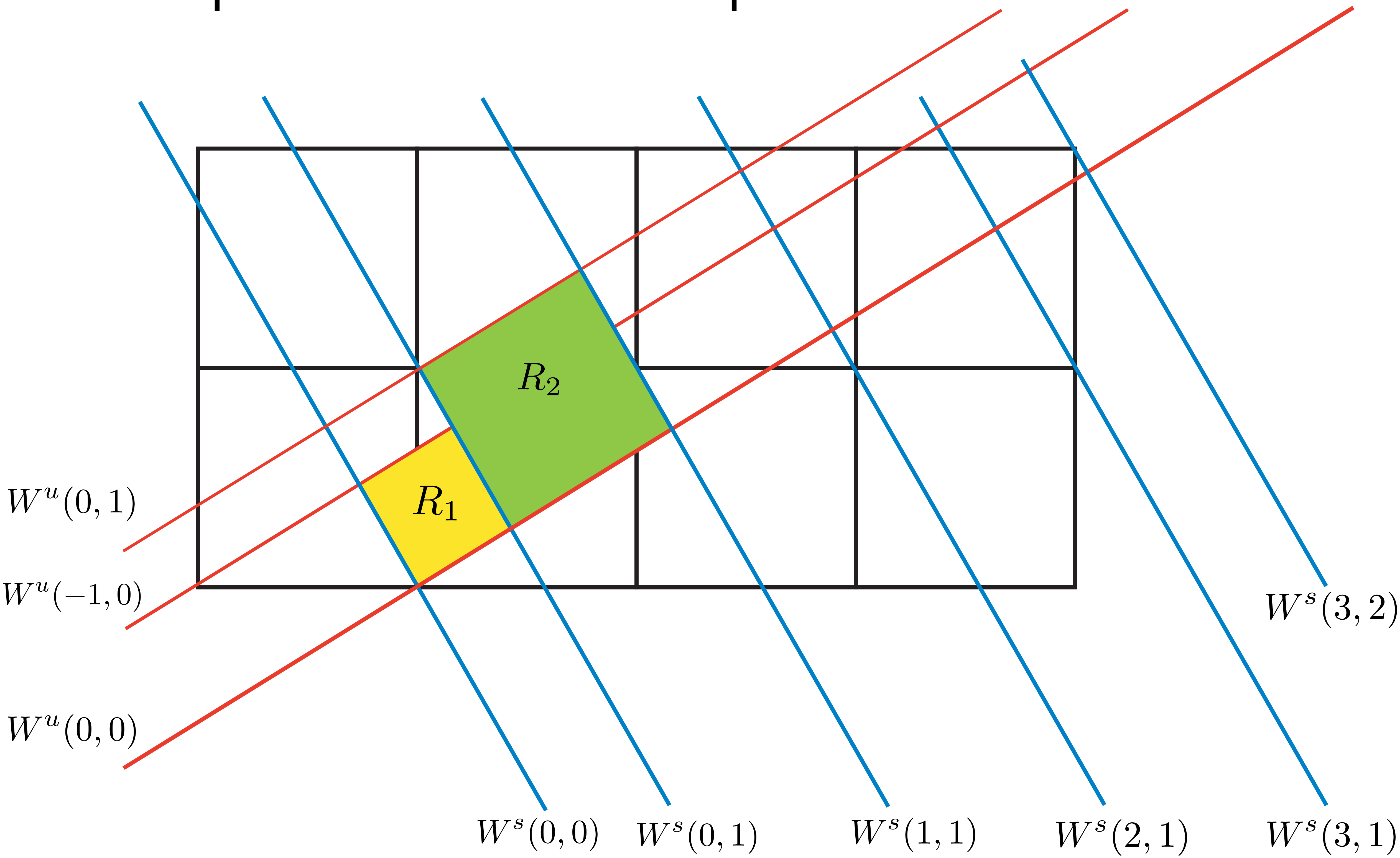
Measurement Theory ...

Markov partition for 2D Cat Map ...

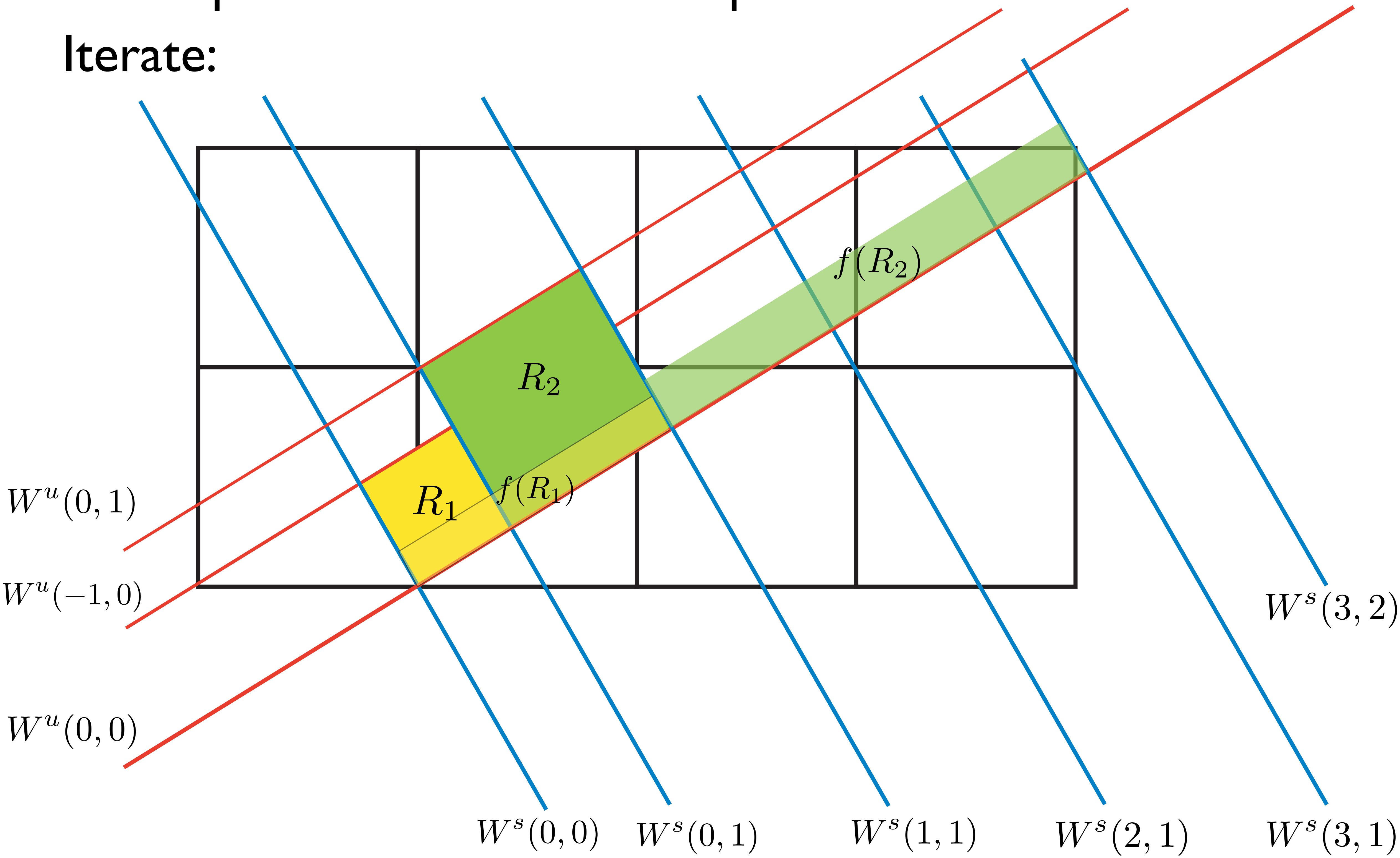
Partition tiles the plane:



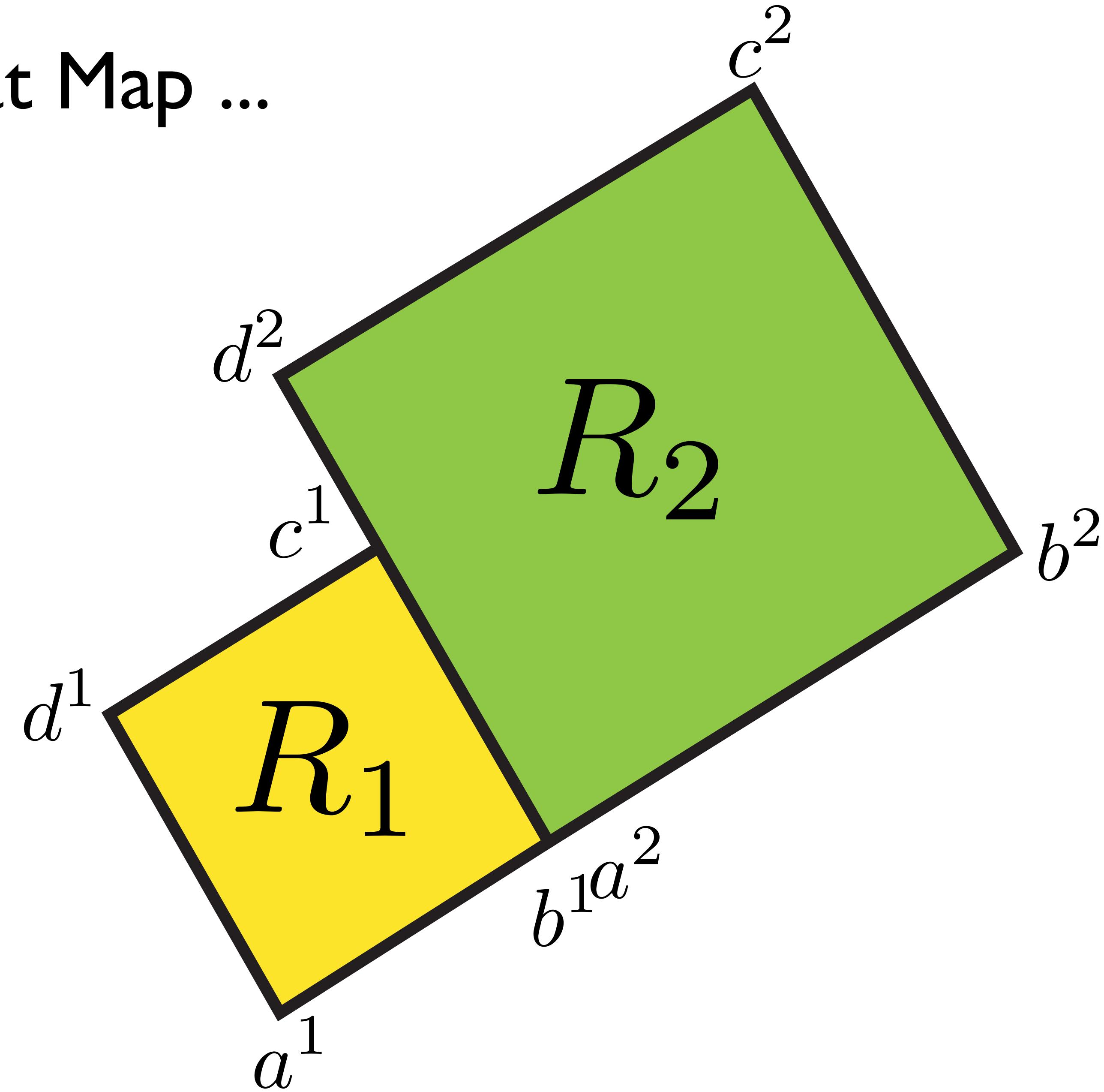
From Determinism to Stochasticity ...
Measurement Theory ...
Markov partition for 2D Cat Map ...



From Determinism to Stochasticity ...
Measurement Theory ...
Markov partition for 2D Cat Map ...
Iterate:



From Determinism to Stochasticity ...
Measurement Theory ...
Markov partition for 2D Cat Map ...



$$\mathcal{P} = \{R_1, R_2\}$$

$$R_1 = \{(0,0), (b_x^1, b_y^1), (c_x^1, c_y^1), (d_x^1, d_y^1)\}$$

$$R_2 = \{(a_x^2, a_y^2), (b_x^2, b_y^2), (c_x^2, c_y^2), (d_x^2, d_y^2)\}$$

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Measurement Theory ...

Markov partition for 2D Cat Map ...

$$(a_x^2, a_y^2) = (b_x^1, b_y^1) = W^s(0,1) \bigcap W^u(0,0)$$

$$= (\tfrac{1}{\sqrt{5}}, \tfrac{5-\sqrt{5}}{10})$$

$$(b_x^2, b_y^2) = W^u(0,0) \bigcap W^s(1,1)$$

$$(c_x^2, c_y^2) = (d_x^1, d_y^1) = W^u(0,1) \bigcap W^s(1,1)$$

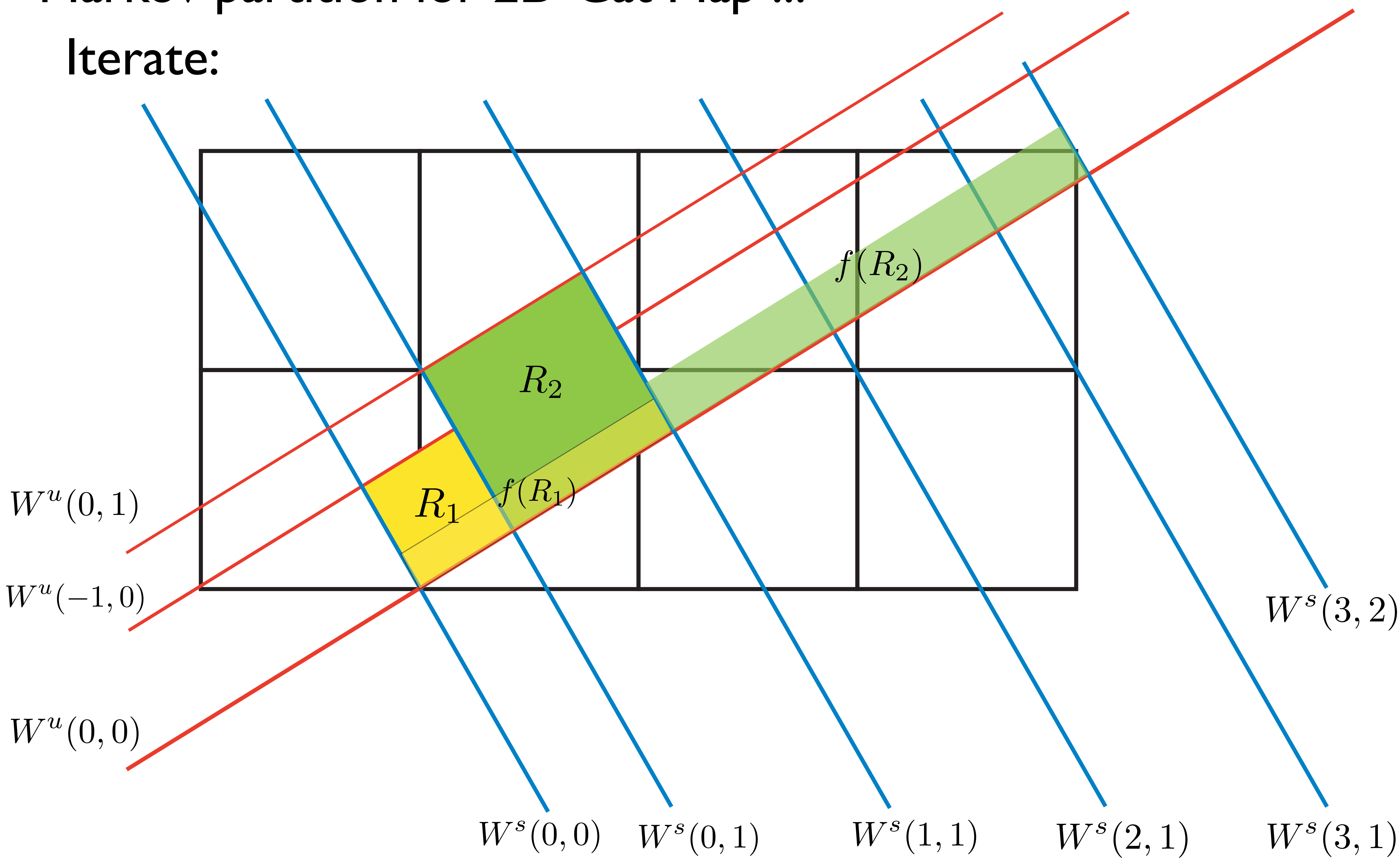
$$(c_x^1, c_y^1) = W^s(0,1) \bigcap W^u(-1,0)$$

From Determinism to Stochasticity ...

Measurement Theory ...

Markov partition for 2D Cat Map ...

Iterate:

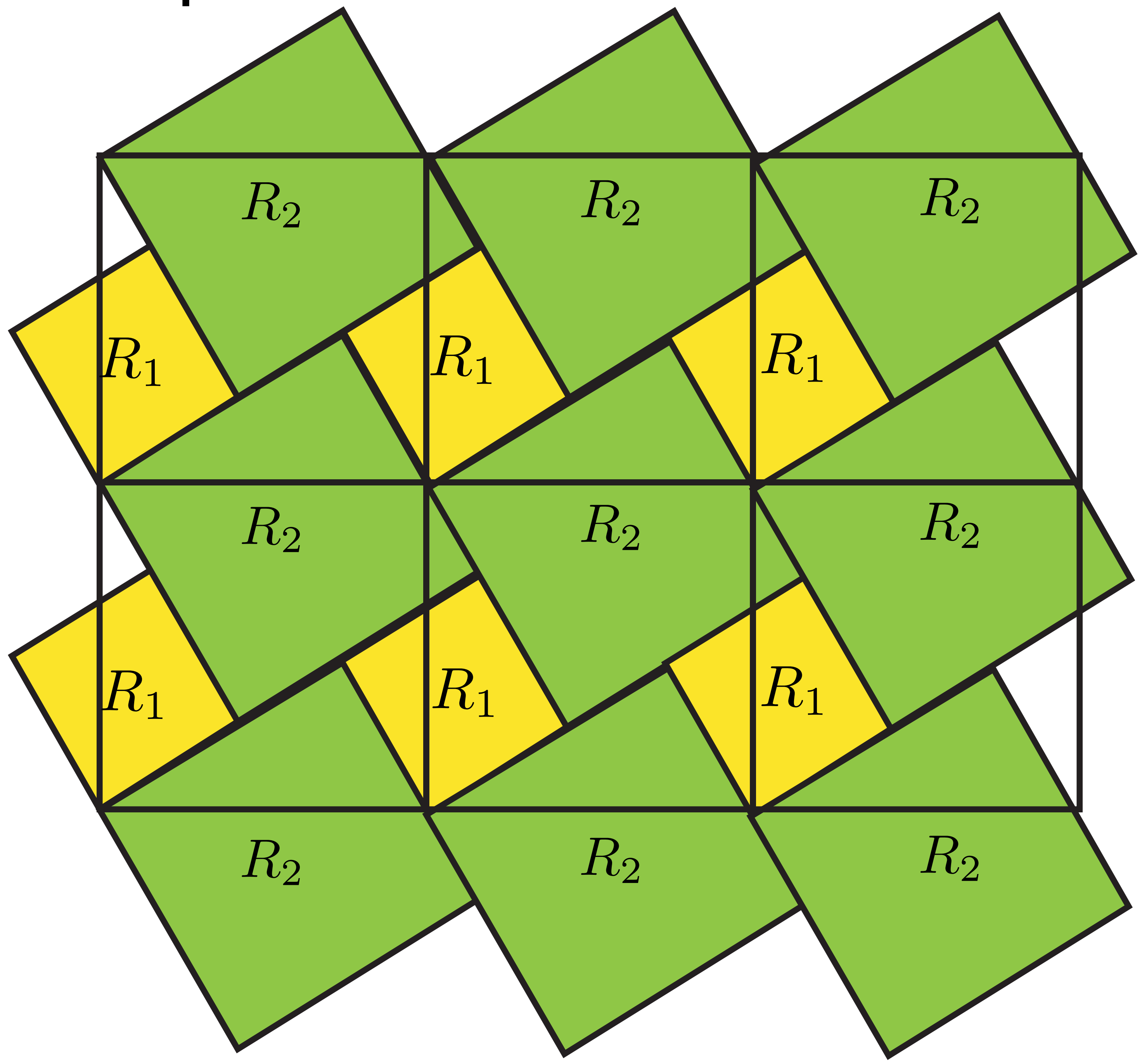


From Determinism to Stochasticity ...

Measurement Theory ...

Markov partition for 2D Cat Map ...

Partition tiles the plane:



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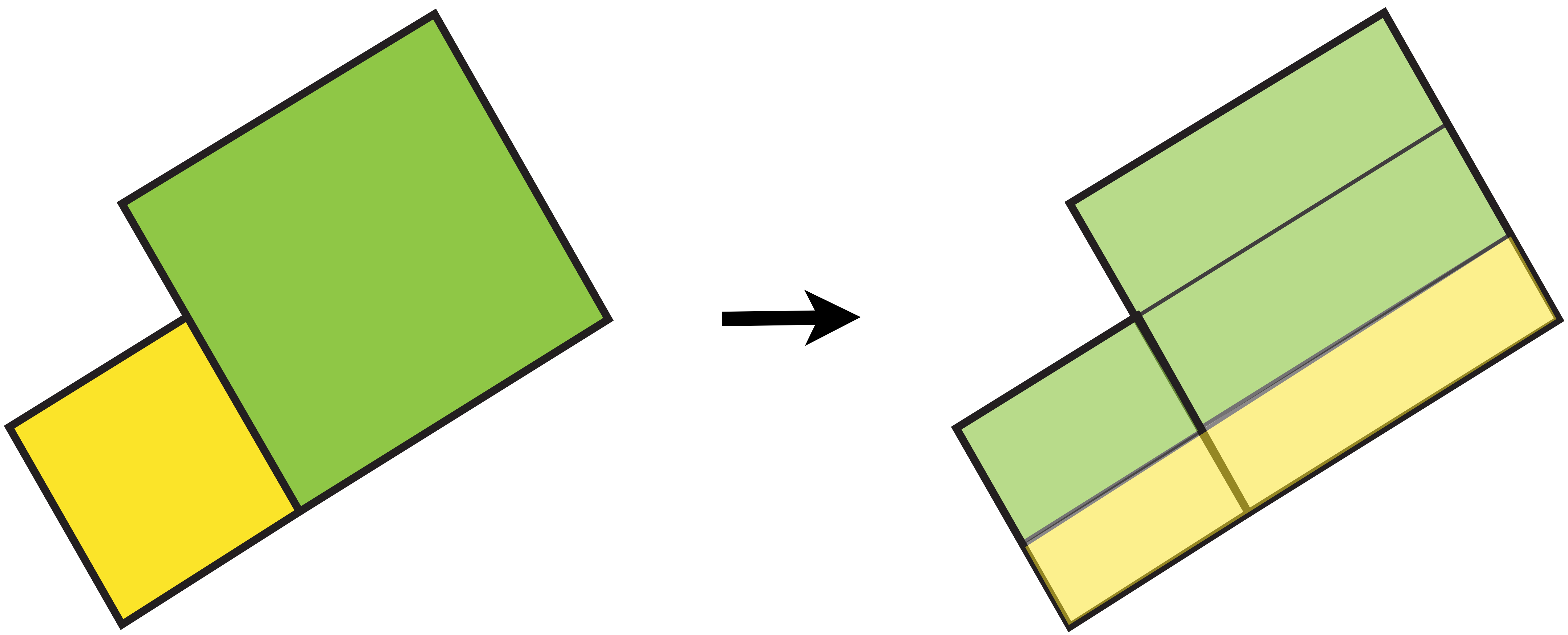
Measurement Theory ...

Markov partition for 2D Cat Map ...

Action of map on partition:

$$f(R_1) \subseteq R_1 \bigcup R_2 \qquad f(R_2) \subseteq R_1 \bigcup R_2$$



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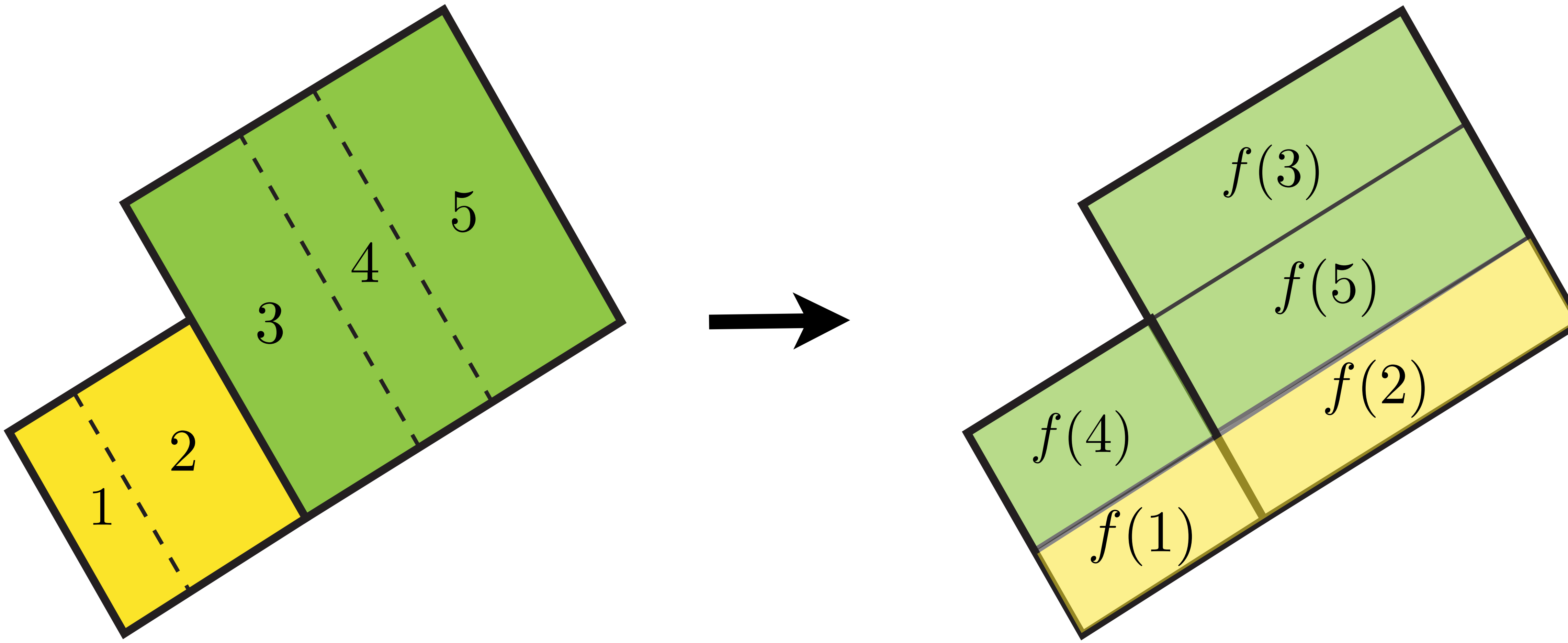
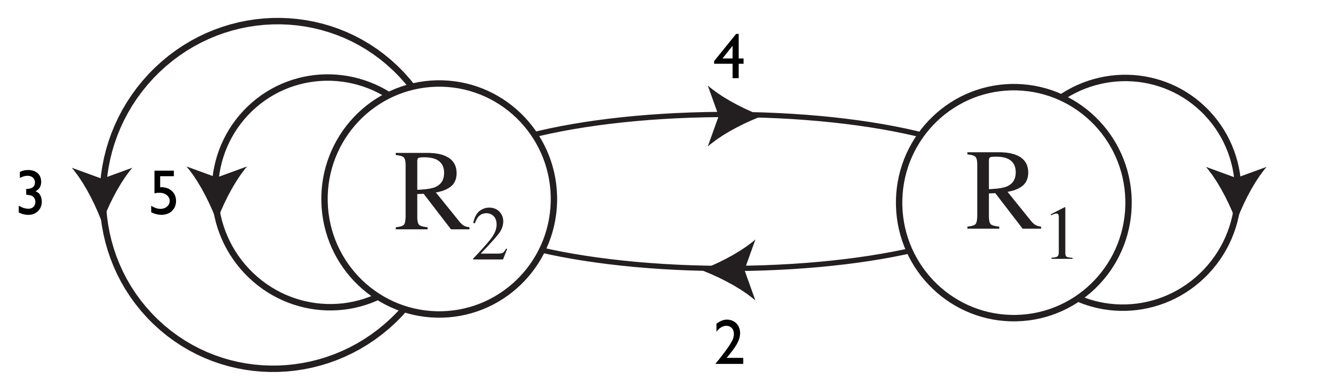
Measurement Theory ...

Markov partition for 2D Cat Map ...

Action of map on partition ...



Markov chain:



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Measurement Theory ...

Markov partition very stringent:

Partition boundaries map to partition boundaries.

Generating partitions: Idea of "good" instrument.

Easier to find than Markov partitions, which may not exist.
Only requirement: Sequences track individual orbits

$$\|\Delta(s^L)\| \to 0, \text{ for all } s^L \in \Sigma_f, \text{ as } L \to \infty$$

Cylinders $\Delta(s^L)$ label points in the state space $x \in M$

Requires chaos:
Instability translates into reverse-time shrinking of cells.

Caveats:
Finite-to-one mapping of sequences to orbits is okay.
Analog: x = "1" is both s = 1.00000... and s = 0.999999....
Ambiguity for points on partition cell boundaries.

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Measurement Theory ...

Some facts about partitions:

Markov partitions are generating.

Markov partition reduces Perron-Frobenius operator to finite-dimensional stochastic transition matrix.

Resulting process is modeled by finite, probabilistic Markov chain.

Generating partition may lead to finite- or infinite-dimensional hidden Markov model.

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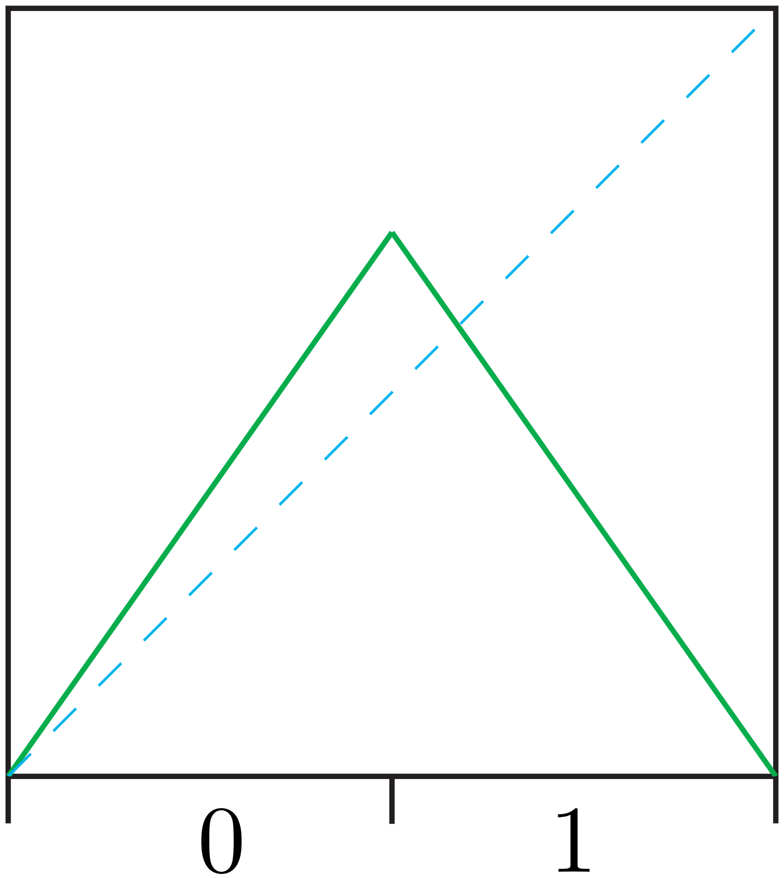
Measurement Theory ...

Generating partition for Tent map:

At any parameter (slope > 1):

$$\mathcal{P} = \{0 \sim [0, \tfrac{1}{2}], 1 \sim [\tfrac{1}{2}, 1]\}$$



Not Markov!
except at slope = 2.

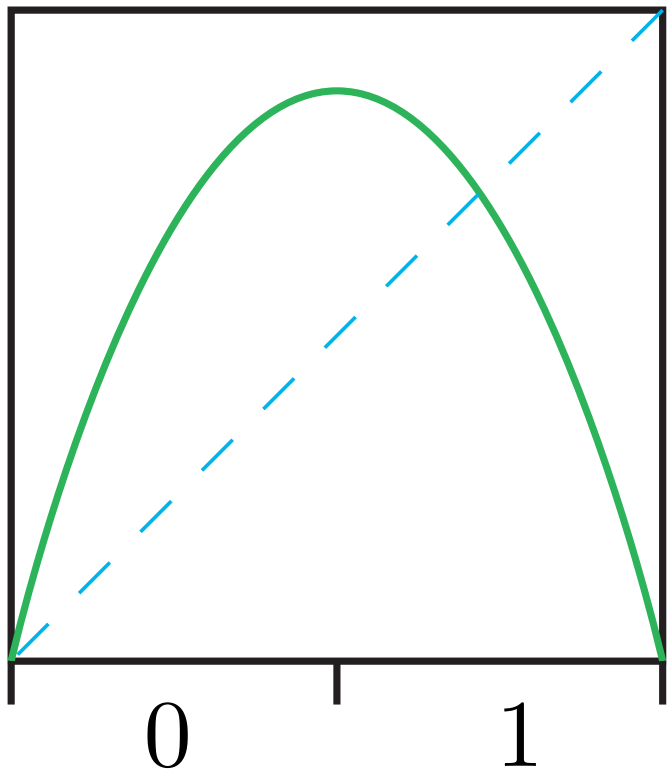
From Determinism to Stochasticity ...

Measurement Theory ...

Generating partition for Logistic map:

At any parameter:

$$\mathcal{P} = \{0 \sim [0, \tfrac{1}{2}], 1 \sim [\tfrac{1}{2}, 1]\}$$



Not Markov!
except at r = 4.

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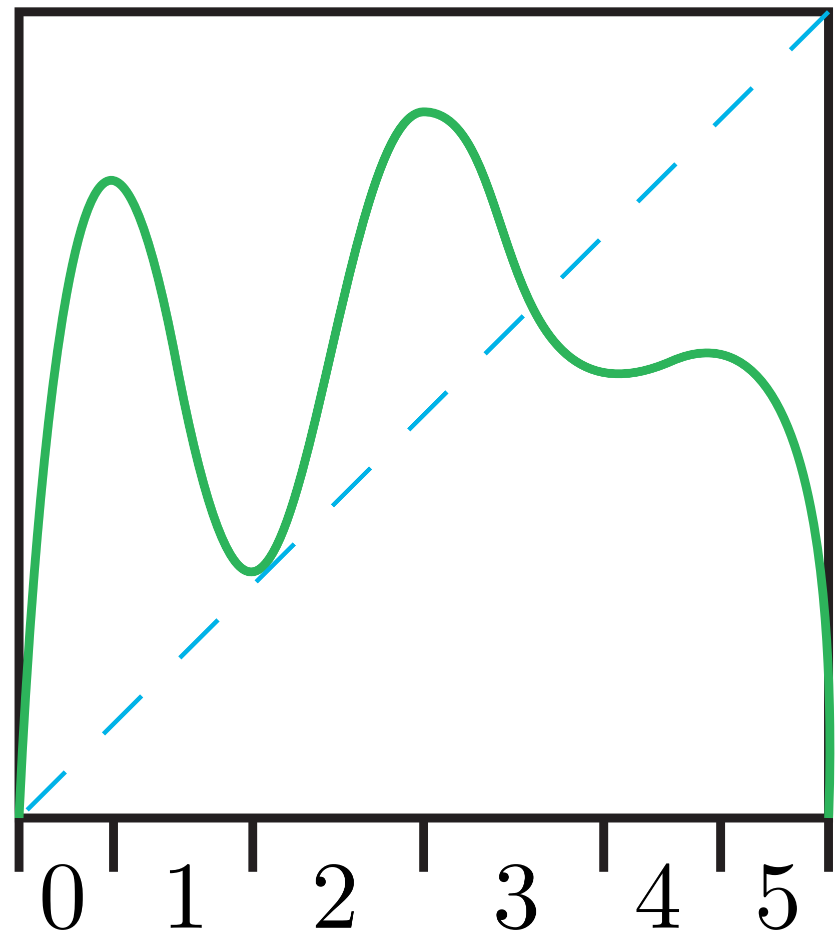
Measurement Theory ...

Generating partition for general 1D maps:

Lap: Monotone piece of $f(x)$

Partition:

$$\mathcal{P} = \{\text{domain of lap}(f)\}$$



Theorem: If map is chaotic, P is generating.

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Measurement Theory ...

What happens when there is

no Markov partition and

no generating partition?

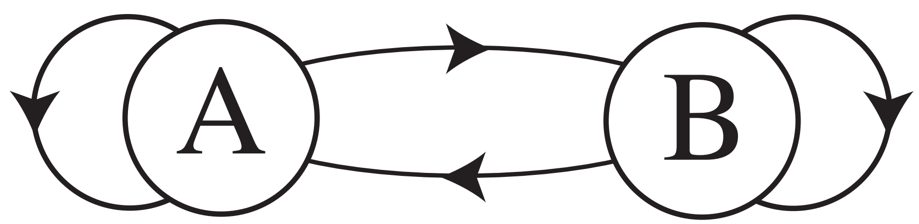
From Determinism to Stochasticity ...

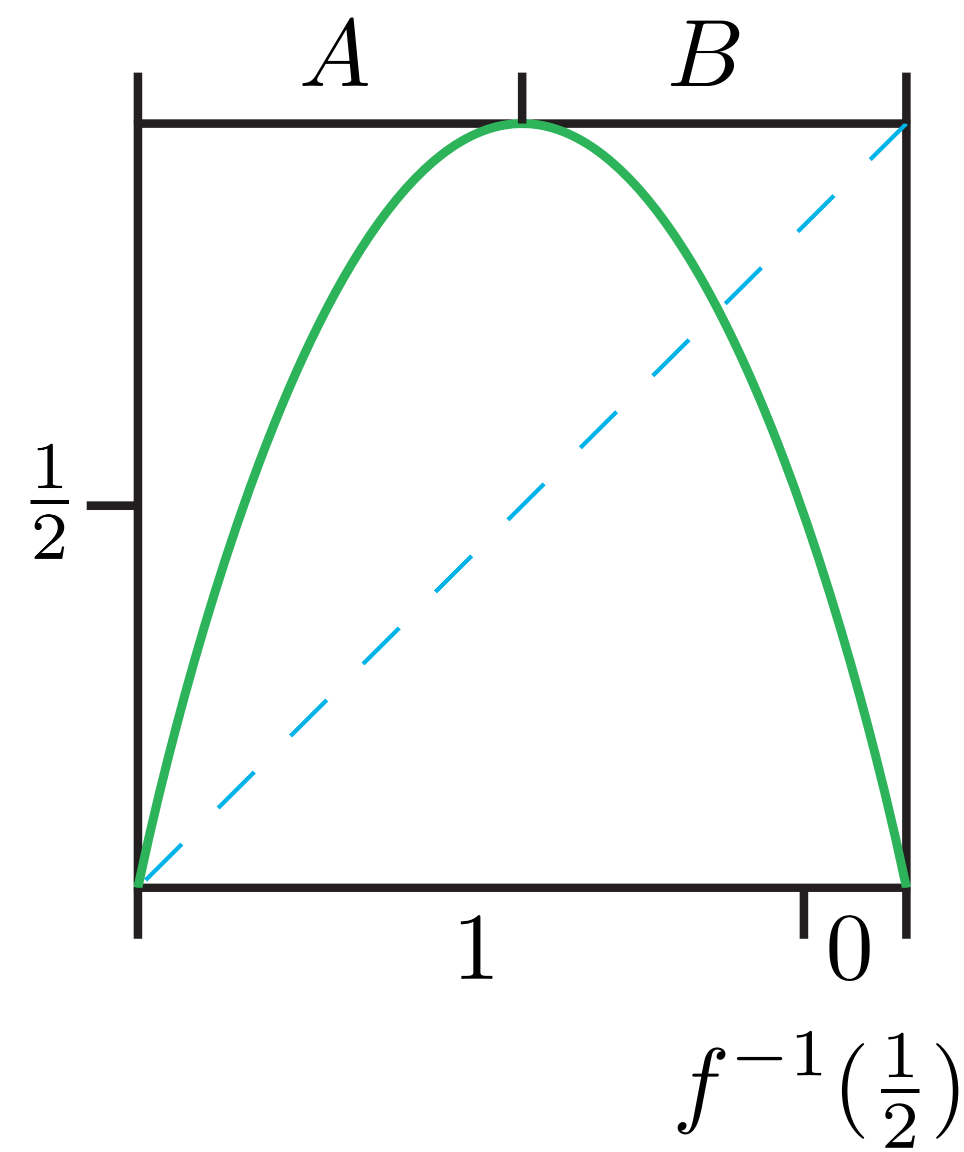
Measurement Theory ...

Example of a nongenerating partition:

Logistic map at 2 onto 1

Internal Markov Model:





Measurement alphabet:

$$\{0, 1\}$$

Measurement partition:

$$\mathcal{P} = \{1 \sim [0, d], 0 \sim (d, 1]\}$$

Decision point:

$$d = \max \{f^{-1}(\tfrac{1}{2})\}$$

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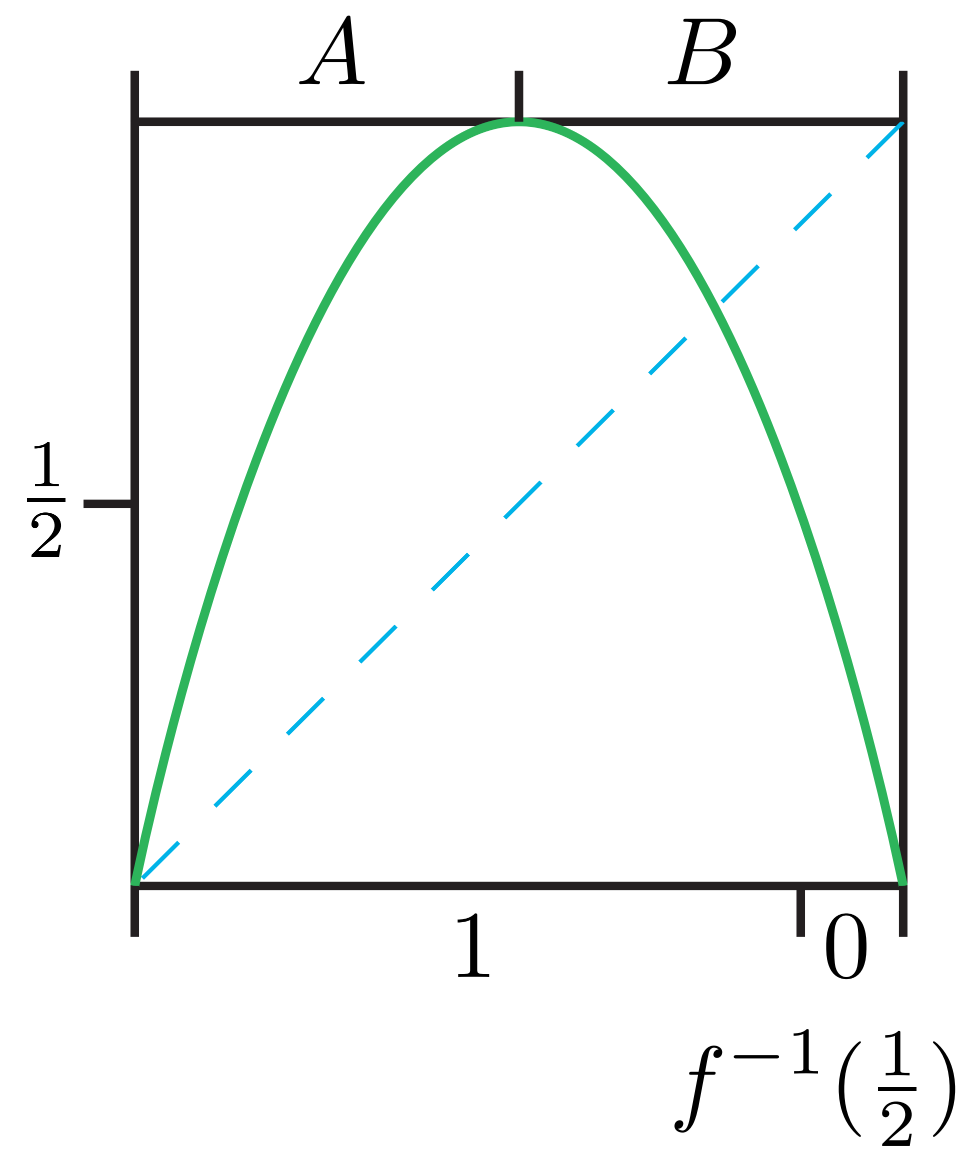
Measurement Theory ...

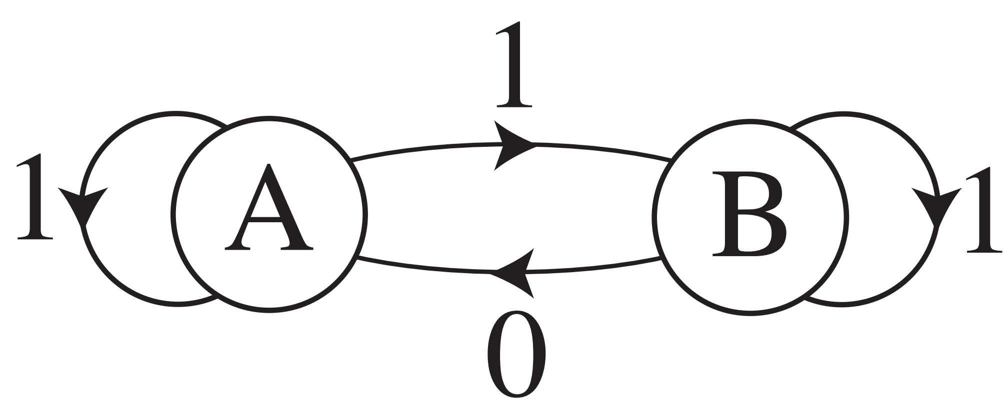
Example of a nongenerating partition ...

Decision point:

$$d = \max\ \{f^{-1}(\tfrac{1}{2})\}$$



Hidden Markov Model:



Nonunifilar!

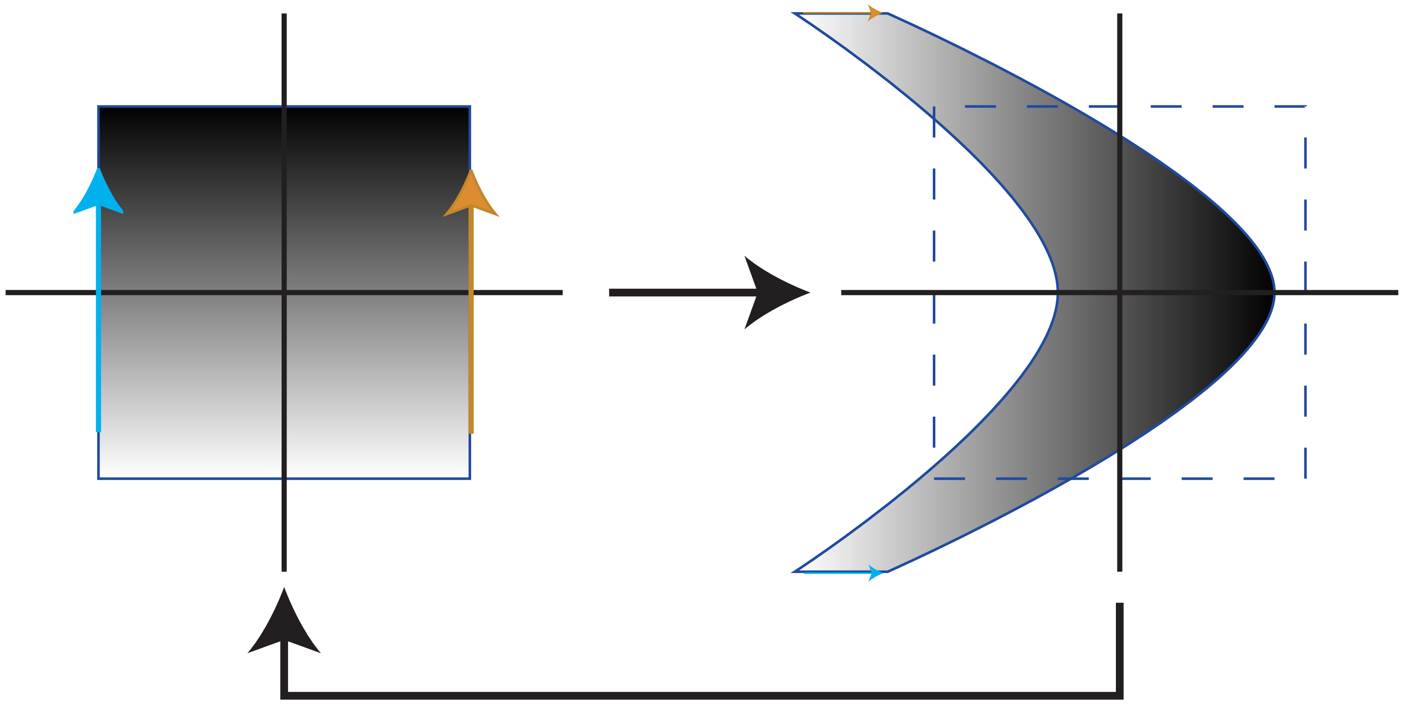
The Simple Nonunifilar Source.

Measurement Theory ...

Almost generating partitions for a 2D Map:

Hénon map $(x, y) \in \mathbf{R}^2$

$$
\begin{aligned}
x_{n+1} &= y_n + 1 - a x_n^2 \\
y_{n+1} &= b x_n
\end{aligned}
$$



Note: Hénon map becomes Logistic map if b = 0.

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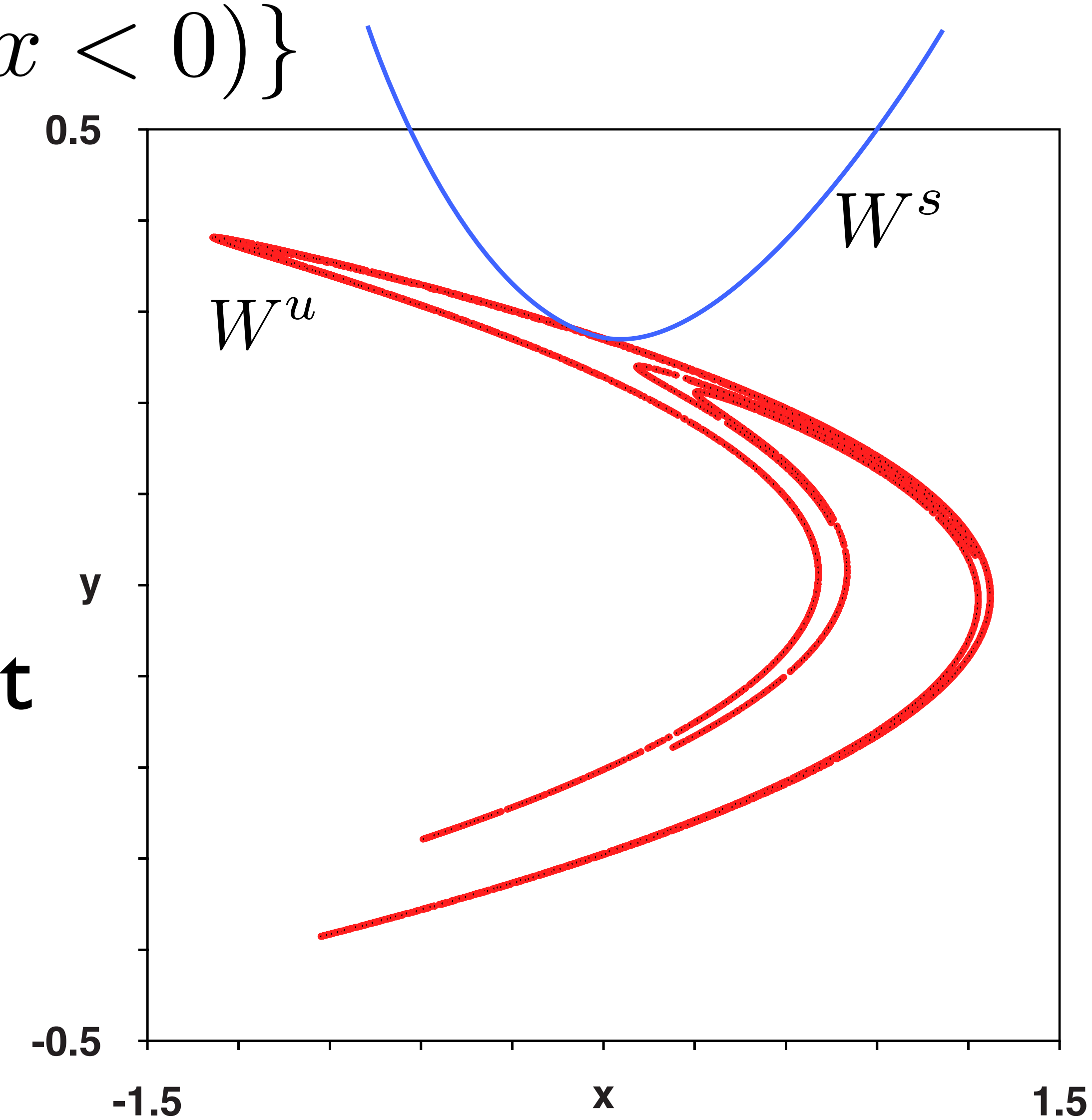
Measurement Theory ...

Almost generating partitions for a 2D Map ...

Hénon map becomes Logistic map if b = 0.

Suggests trying:

$$\mathcal{P} = \{1 \sim x \in (x > 0), 0 \sim (x < 0)\}$$



Not generating!

Due to failure of hyperbolicity.

Stable & unstable manifold tangent at some points.

From Determinism to Stochasticity ...

Measurement Theory ...

Synopsis:

How to model measurement process.

How chaos interacts with measurement partition.

End up with discrete-valued sequences.

What kind of stochastic process is the result?

Markov process? Sometimes.
Hidden Markov process? Sometimes.
Sometimes finite, sometimes infinite order.

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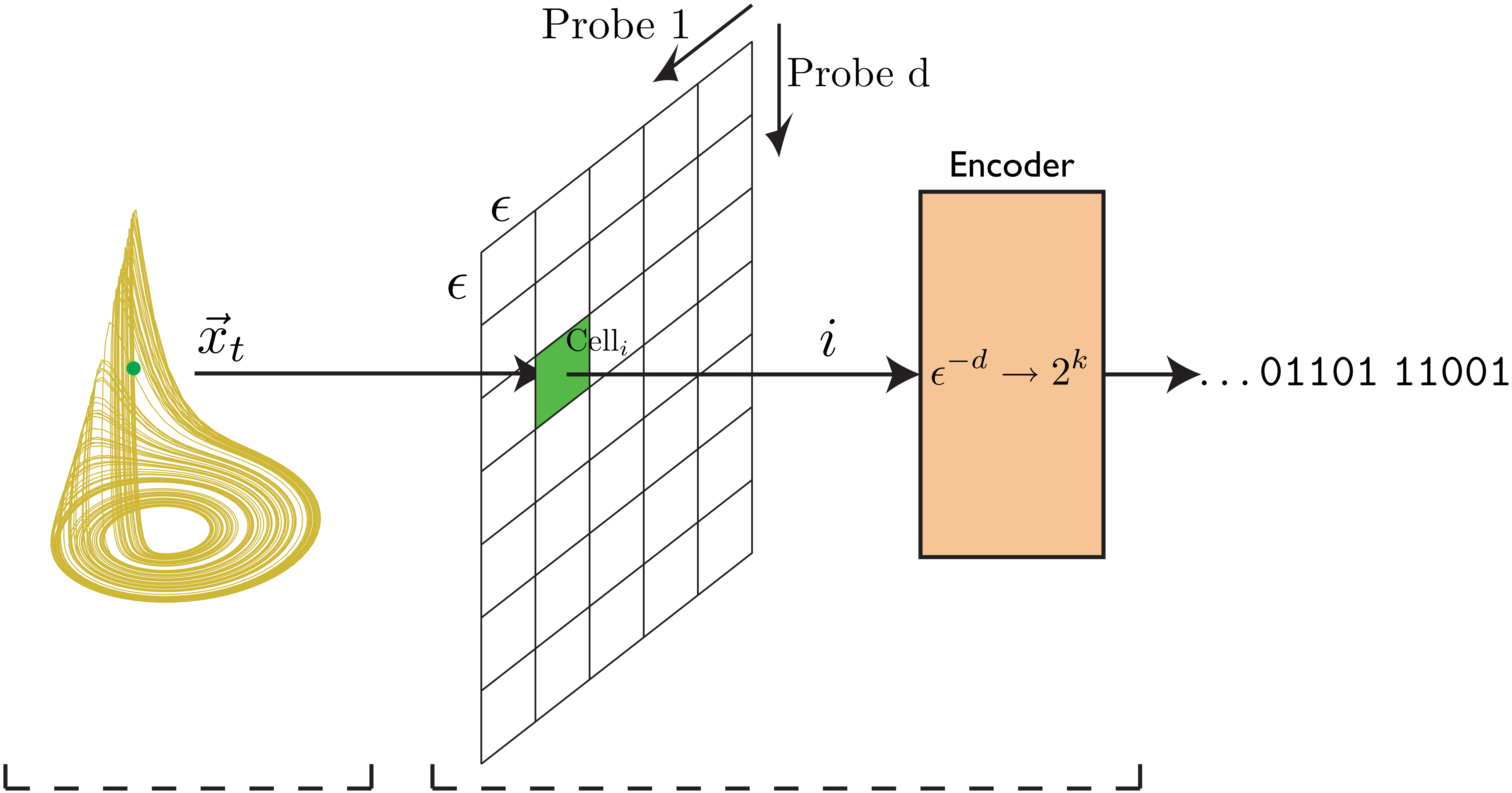

Measurement Channel

From Determinism to Stochasticity ...

Measurement Theory ...

Main questions now:

How do we characterize the resulting process?

Measure degrees of unpredictability & randomness.

What correlational structure is there?

How do we build a model from the process itself?

How much can we reconstruct about the hidden internal dynamics?

From Determinism to Stochasticity ...

Reading for next lecture:

Elements of Information Theory (EIT), Chapters 1 & Sections 2.1-2.8.