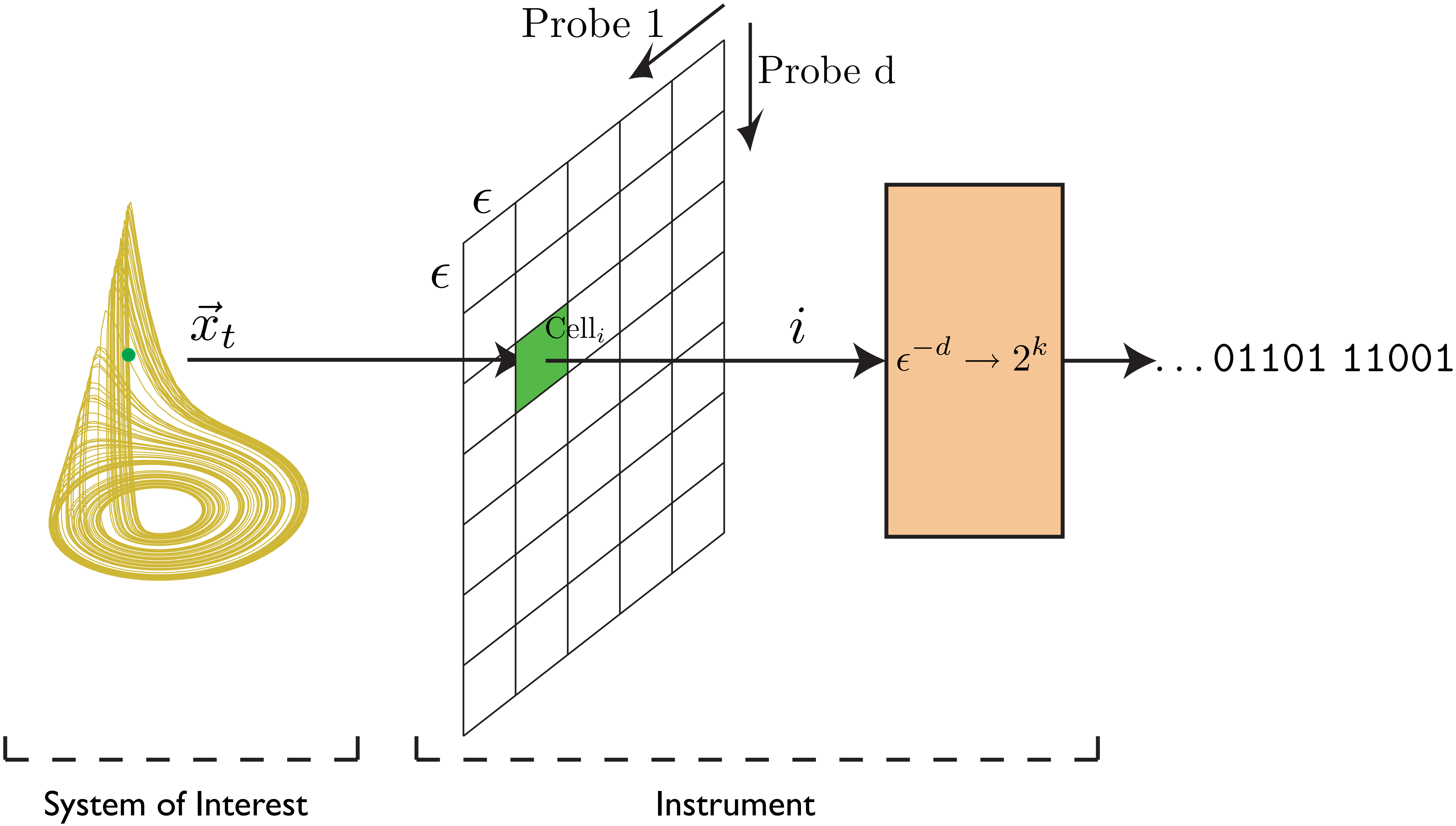
From Determinism to Stochasticity
Measurement Theory I

Reading for this lecture:

(These) *Lecture Notes*.

CMPy Interactive Labs:
Symbolic Dynamics and Partitions

From Determinism to Stochasticity ...



Measurement Channel

From Determinism to Stochasticity ...

Measurement Theory: Making the connection

Hidden Dynamical System:

What can we learn from discrete time series?

Know how to evolve:

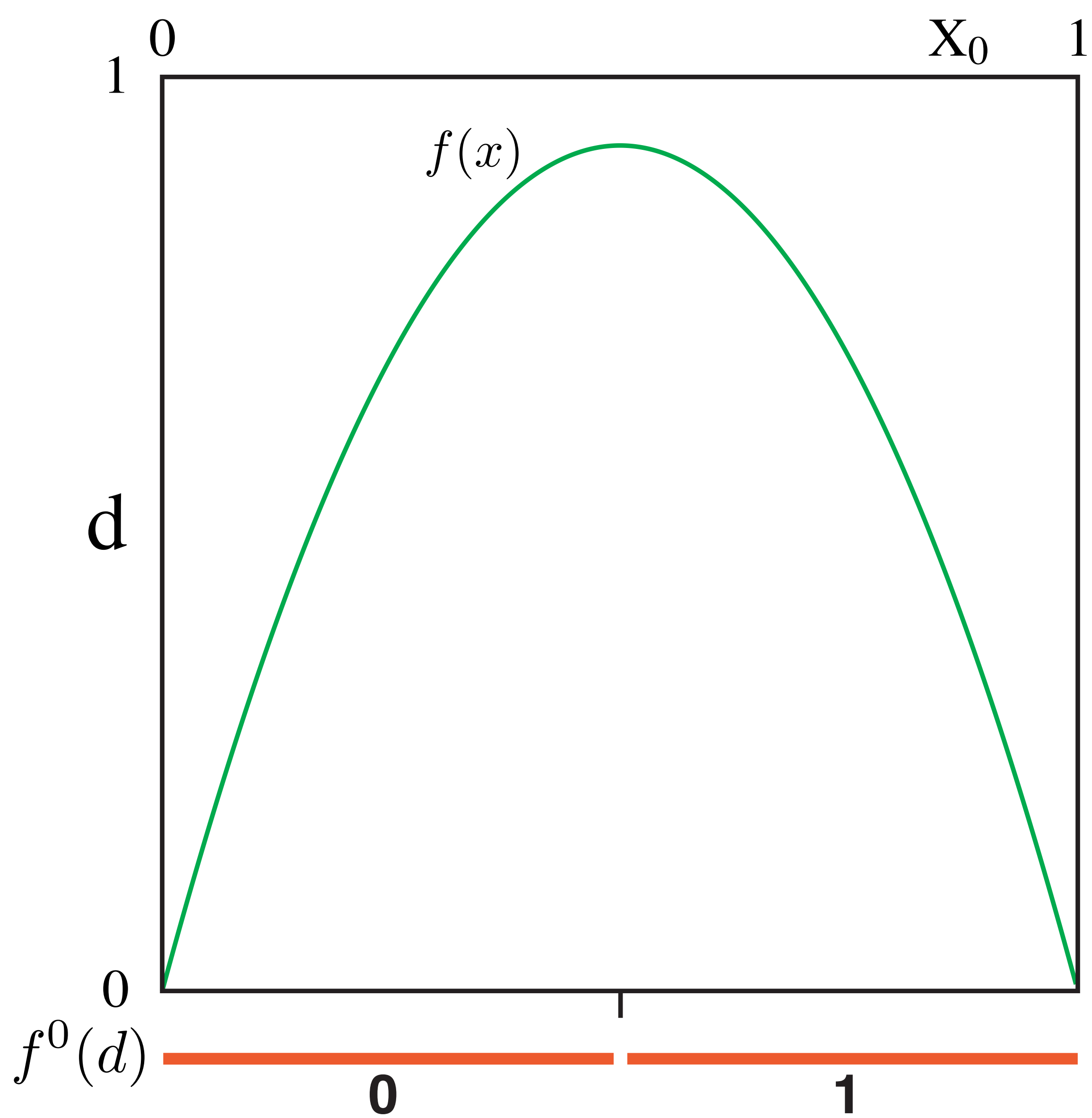
$$\mathcal{T} : \vec{x}_0 \rightarrow \vec{x}_1$$

$$\mathcal{T} : p_0(x) \rightarrow p_1(x)$$

How to evolve boundaries?

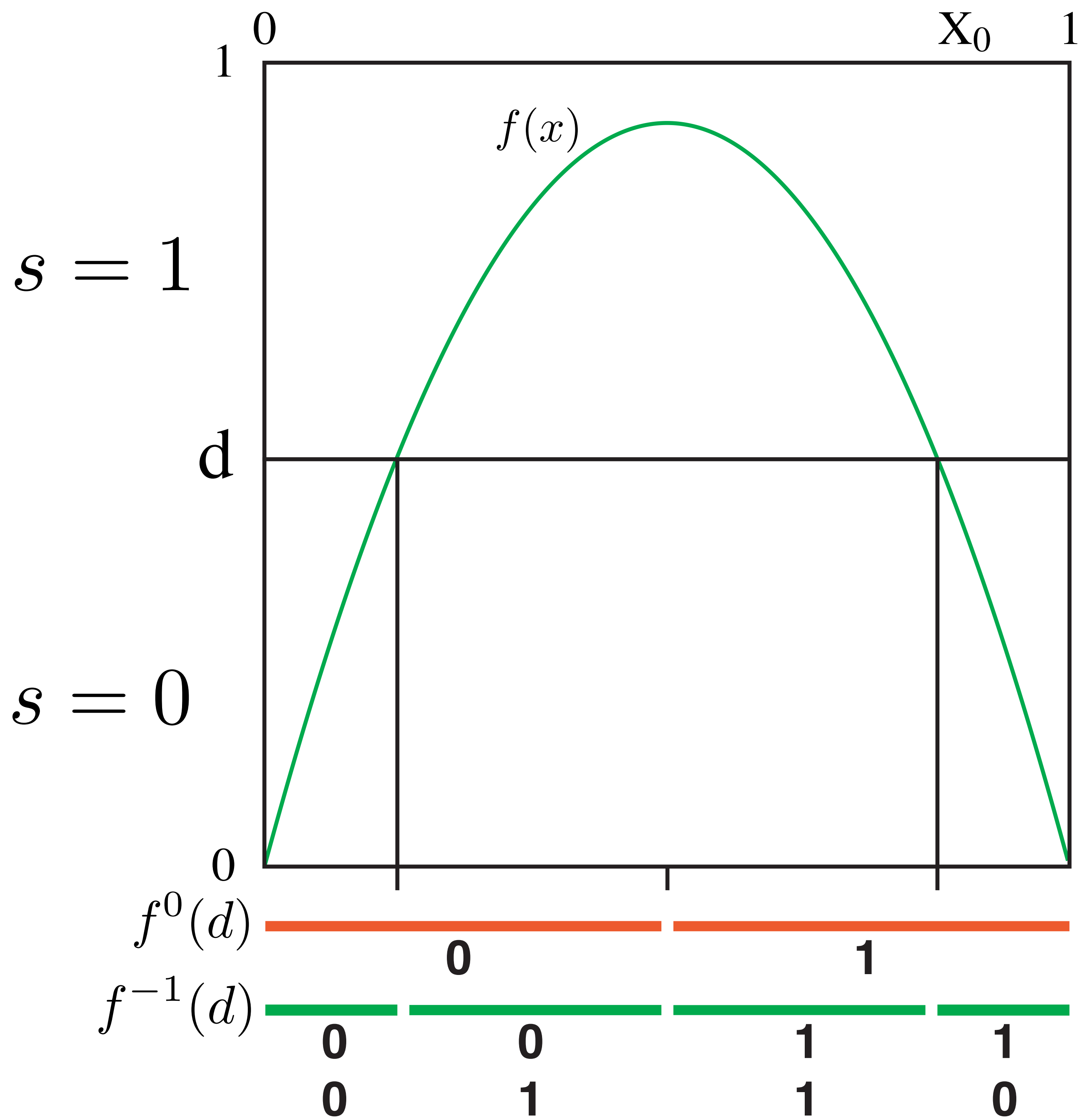
From Determinism to Stochasticity ...

Measurement Theory of 1D Maps ...



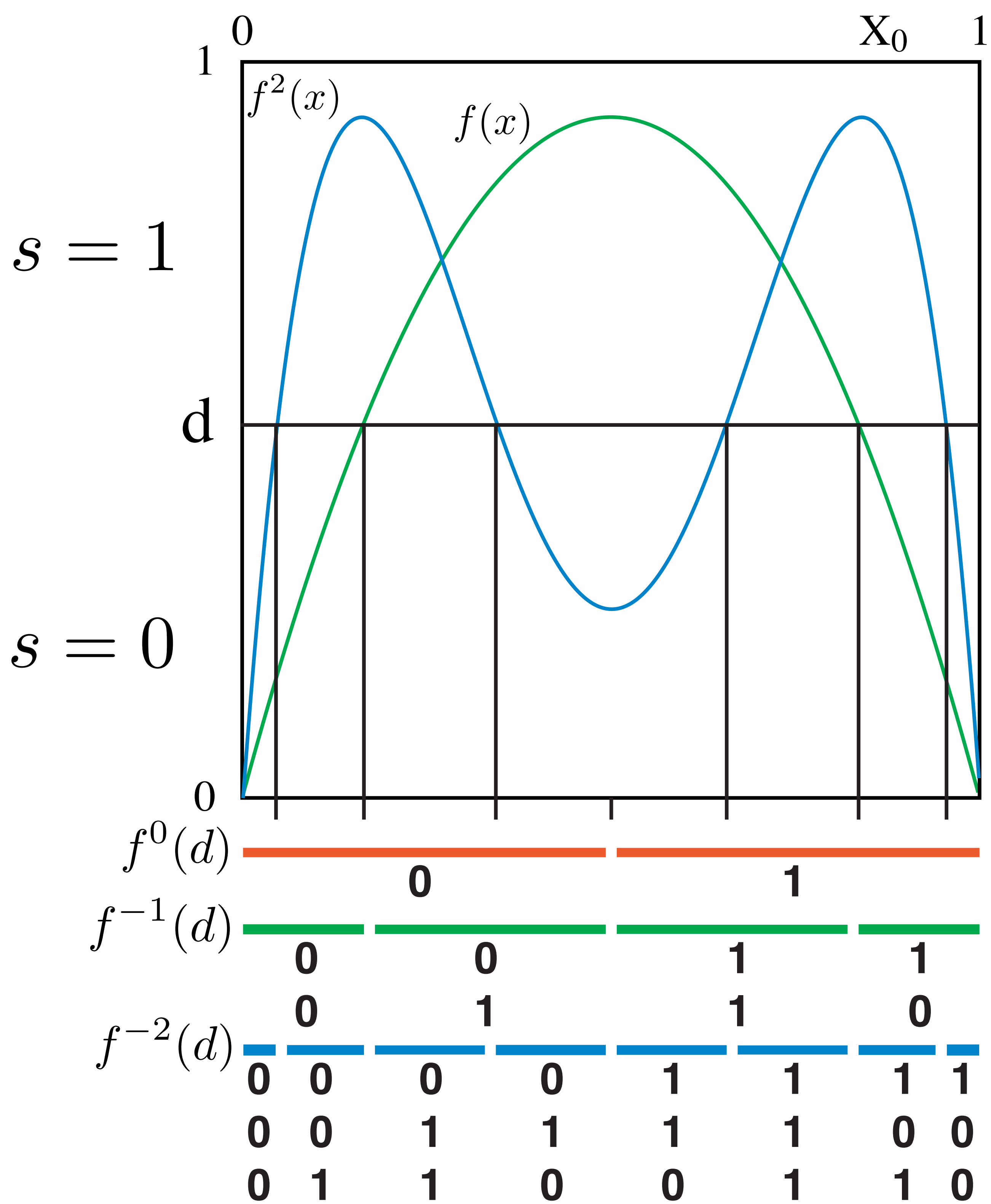
From Determinism to Stochasticity ...

Measurement Theory of 1D Maps ...



From Determinism to Stochasticity ...

Measurement Theory of 1D Maps ...



From Determinism to Stochasticity ...

Measurement Theory ...

Dynamical System: $\{f, M\}$

$$f : M \to M$$

Partition:

$$\mathcal{P} = \{P_i \subset M : i = 1, \ldots, p\}$$

$$M = \bigcup_{i=1}^{p} P_i$$

$$P_i \cap P_j = \emptyset, \ i \neq j$$

Two Partitions: $\mathcal{P}$ & $\mathcal{Q}$

Refinement: $\mathcal{P} \vee \mathcal{Q} = \{P_i \cap Q_j : P_i \in \mathcal{P} \ \& \ Q_j \in \mathcal{Q}\}$

This is a partition, too.

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Measurement Theory ...

Partition: $\mathcal{P}$

Measurement symbols: $\text{Label}(P_i) = s \in \mathcal{A}$

Measurement operator: $\pi(x) = s, \ x \in P_s$

Orbit: $\mathbf{x} = \{x_0, x_1, x_2, \ldots\}$

Measurement sequence:

$$\begin{aligned} \mathbf{s} &= \pi(\mathbf{x}) \\ &= \{s_0, s_1, s_2, \ldots : s_i = \pi(x_i)\} \end{aligned}$$

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Measurement Theory ...

Orbit Space: $\cdots \times M \times M \times M \times \cdots$

Sequence Space: $$\Sigma = \cdots \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \cdots$$
$$= \{\mathbf{s} = (\ldots s_{-1} s_0 s_1 s_2 \ldots), s_i \in \mathcal{A}\}$$

Dynamic over state space: $f : M \rightarrow M$

Dynamic over sequence space: Shift operator

$$\sigma(\mathbf{s}) = \mathbf{s}'$$

$$s'_i = (\sigma(\mathbf{s}))_i = s_{i-1}$$

Trivial dynamics: All structure now in sequences!

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Measurement Theory ...

Admissible sequences $\Sigma_f \, : \;\; f^i(x_0) \in P_{s_i}$

$$\Sigma_f \subseteq \Sigma$$

$x_0 \in \Lambda$ f-invariant set, then Σ_f is a closed, shift-invariant set:

$$\Sigma_f = \sigma(\Sigma_f)$$

Symbolic Dynamical System under partition $\mathcal{P}$:

Subshift: $\{\Sigma_f , \sigma_f\}$

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Measurement Theory ...

Projection Operator: $\Delta(\mathbf{s}) \subset M$

$$\Delta(\mathbf{s}) = \bigcap_{i=0}^{\infty} f^{-i}(P_{s_i})$$

Admissible: $s^L = s_0 \ldots s_{L-1}$

Sequences that are close to s^L : L-cylinder

$$\mathbf{s}^L = \{\mathbf{s} : s_i = (s^L)_i, i = 0, 1, \ldots, L-1\}$$

Initial conditions whose orbit stays close to $x_0, x_1, x_2, \ldots, x_{L-1}$

$$\Delta(\mathbf{s}^L)$$

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L-cylinder-induced partition of M:

$$\Delta(\mathbf{s}^L) = \left\{ \bigcap_{i=0}^{L} f^{-i}(P_{s_i}) : \;\; s^L = s_0 \ldots s_{L-1} \in \Sigma_f \right\}$$

L-refinement of partition:

$$\mathcal{P}^L = \mathcal{P} \vee f^{-1}\mathcal{P} \vee \cdots \vee f^{L-1}\mathcal{P}$$

$$\Delta(\mathbf{s}^L) \in \mathcal{P}^L$$

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Measurement Theory ...

Symbolic dynamics:

1. Replace complicated dynamic (f) with trivial dynamic (σ)

2. Replace infinitely precise point $x \in M$
   with discrete infinite sequence $\mathbf{s} \in \Sigma_f$

3. If the partition is "good" then
   a. Study discrete sequences to learn about continuous system
   b. Can often calculate quantities directly

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Measurement Theory of 1D Maps:

$$x_{n+1} = f(x_n) \qquad x \in [0,1]$$

$f(x)$ with two monotone pieces

Binary partition: $\mathcal{P} = \{0 \sim x \in [0,d], 1 \sim x \in (d,1]\}$

Decision point: $d \in [0,1]$

L-cylinder induced partition dividers:

$$\mathcal{P}^L = \{d, f^{-1}(d), f^{-2}(d), \ldots, f^{-(L-1)}(d)\}$$

Metrize states:

$$\phi(x) = \sum_{i=0}^{\infty} \frac{\pi(f^i(x))}{2^i}$$

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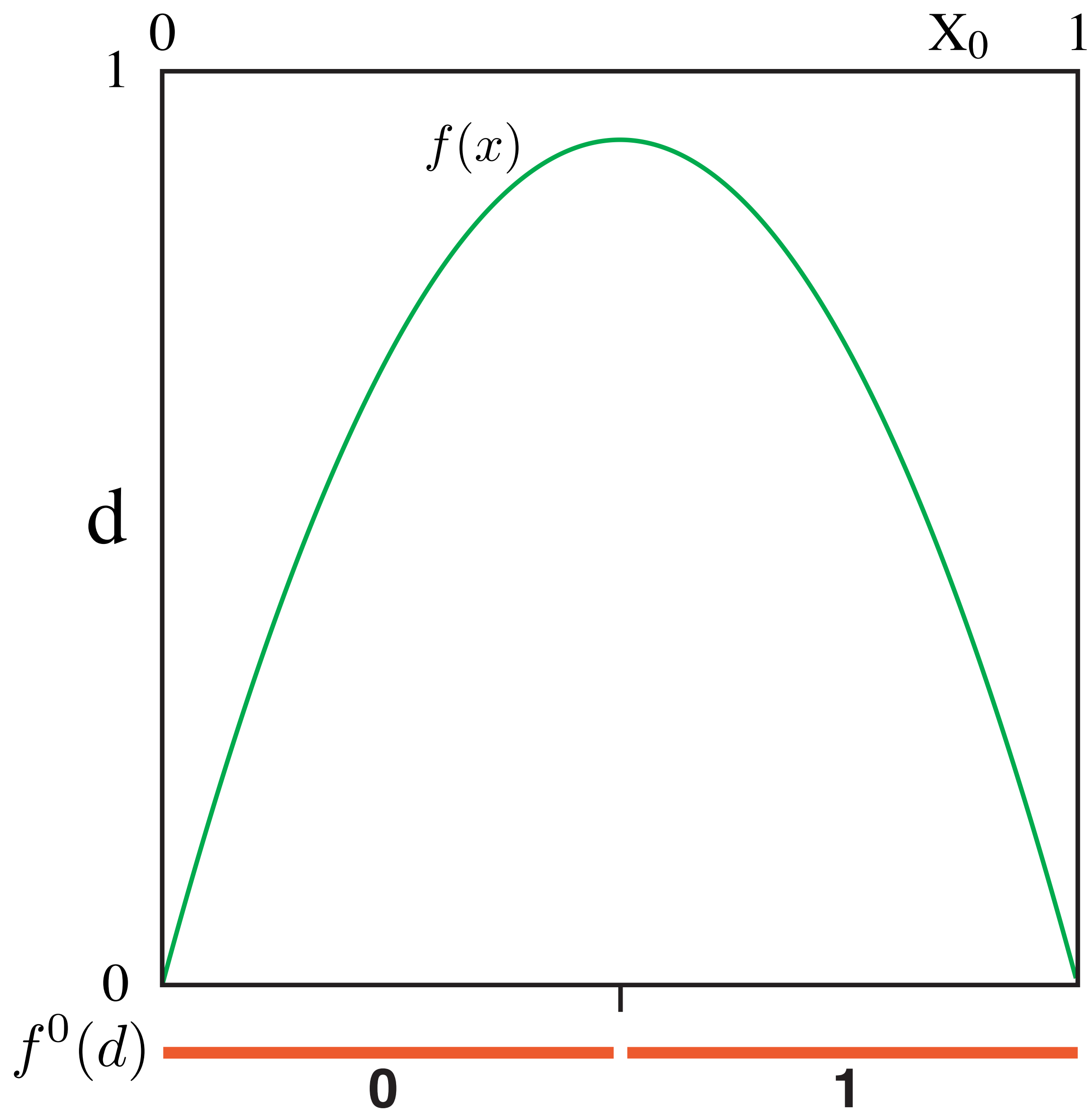
Measurement Theory of 1D Maps ...



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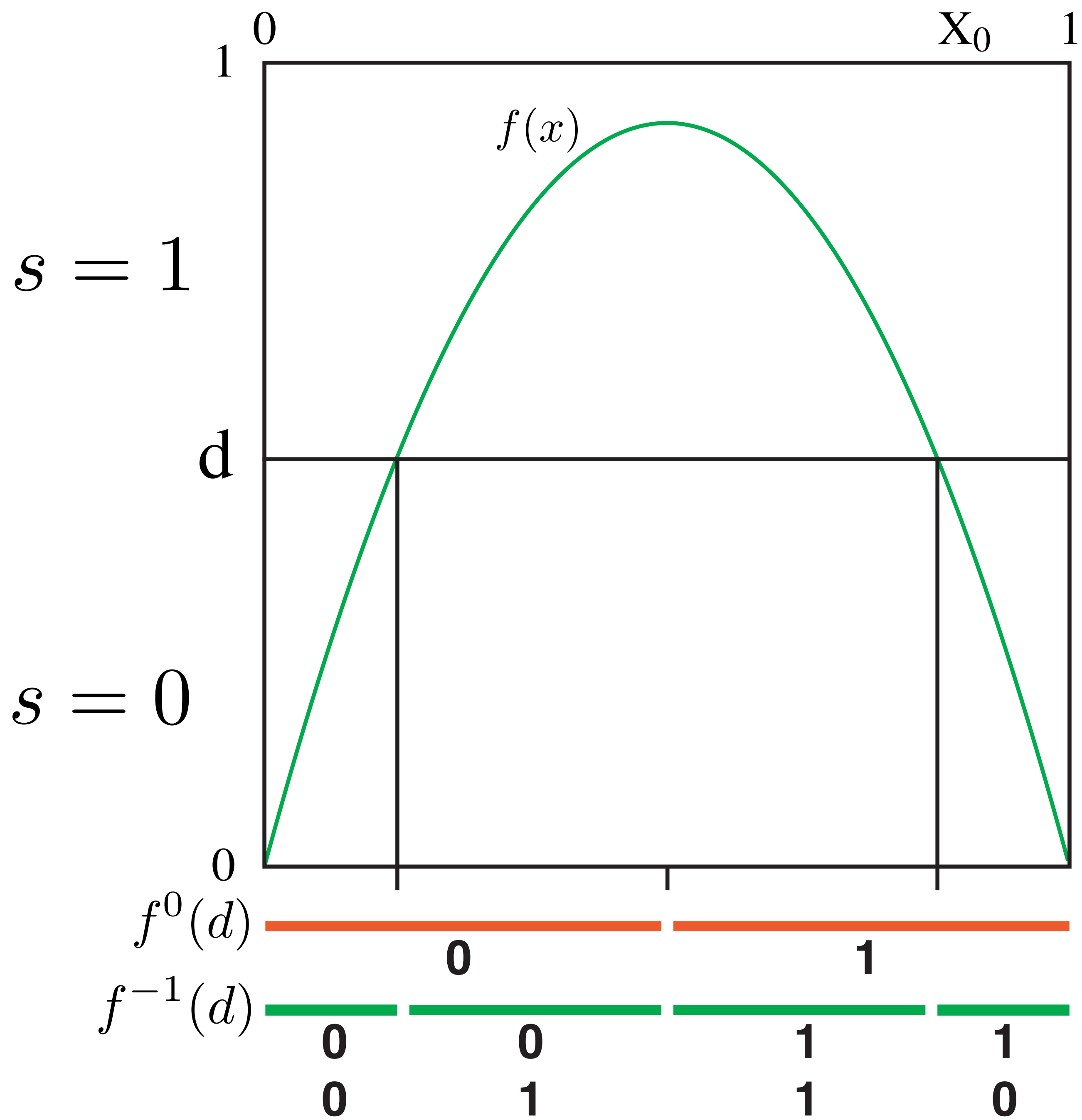
Measurement Theory of 1D Maps ...



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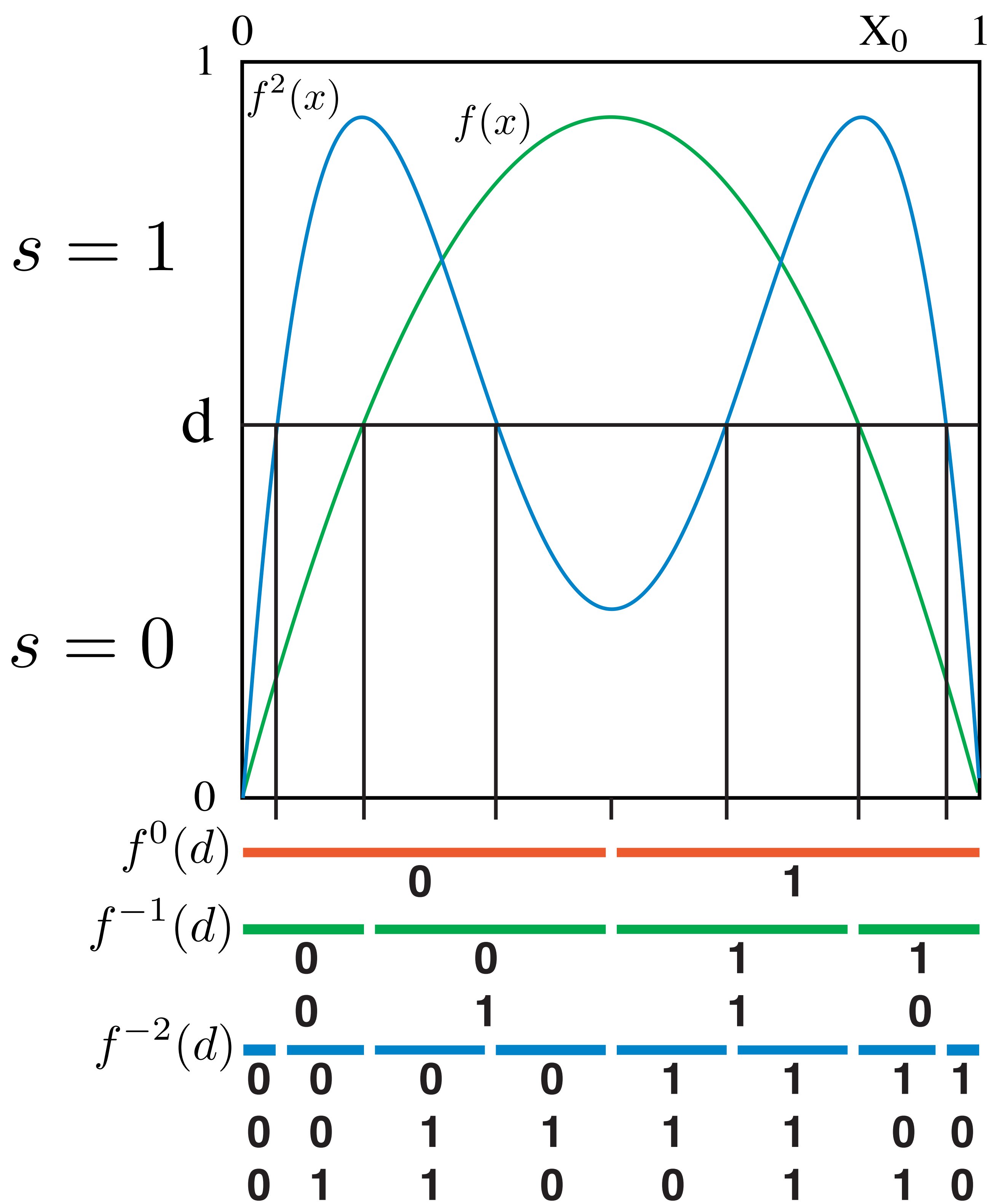

From Determinism to Stochasticity ...

Kinds of Instruments:

When are partitions good?
When symbol sequences encode orbits

$$\begin{array}{ccc} M & \xrightarrow{\mathcal{T}} & M \\ \Delta \uparrow & & \Delta \uparrow \\ \mathcal{A}^{\mathbb{Z}} & \xrightarrow{\sigma} & \mathcal{A}^{\mathbb{Z}} \end{array}$$

Diagram commutes:

$$\mathcal{T}(x) = \Delta \circ \sigma \circ \Delta^{-1}(x)$$

Good kinds of instruments:
- Markov partitions
- Generating partitions

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Measurement Theory ...

Markov Partitions for 1D Maps:

Discrete symbol sequences: $\overleftrightarrow{s} = \overleftarrow{s}\,\overrightarrow{s},\ s \in \mathcal{A}$

Markov = Given symbol, ignore history

$$\Pr(\overrightarrow{s} \mid \overleftarrow{s}) = \Pr(\overrightarrow{s} \mid s_1)$$

Maps of the interval?

$$f : I \rightarrow I,\ I = [0, 1]$$

Partition: $\mathcal{P} = \{P_1, \ldots, P_p\}$

Open sets: $P_i = (d_{i-1}, d_i),\ 0 = d_0 < d_1 < \cdots < d_p = 1$

$$I = \bigcup_{i=1}^{p} \bar{P}_i$$

Markov Partitions for 1D Maps ...

$\mathcal{P}$ is a Markov partition for f :

$$f(P_i) = \bigcup_j P_j, \forall i$$

$f(P_i)$ is 1-to-1 and onto (homeomorphism)

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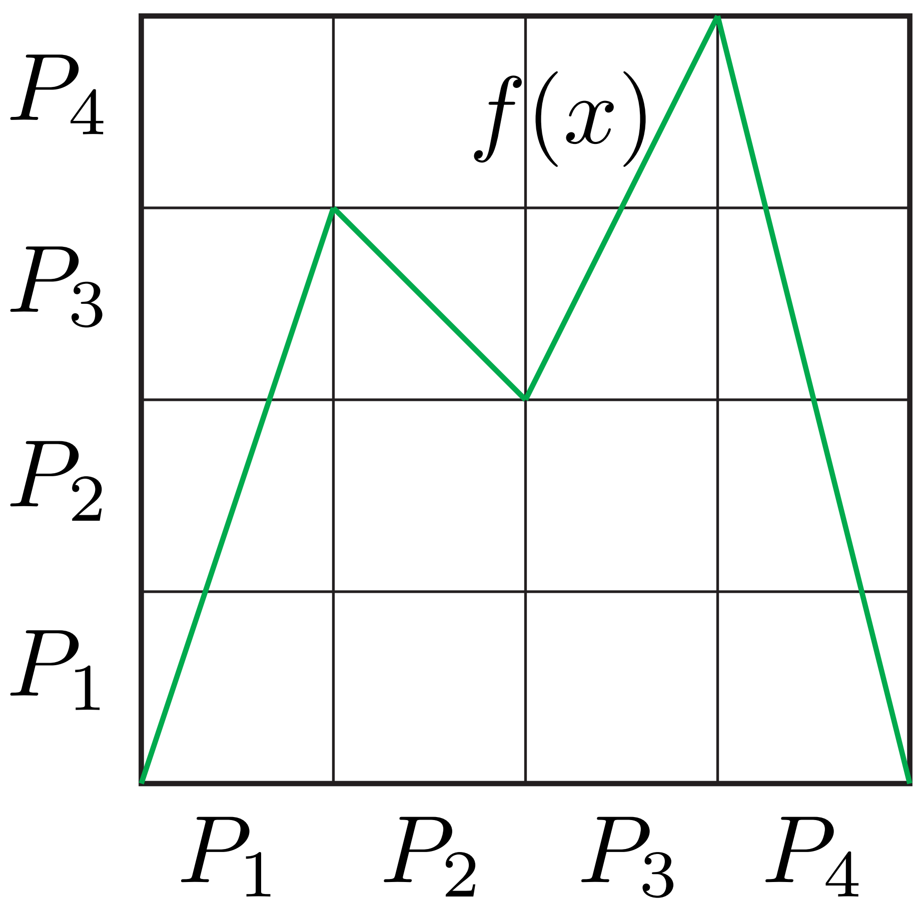
Measurement Theory ...

Markov Partitions for 1D Maps ...

$$s \in \mathcal{A} = \{1, 2, 3, 4\}$$



$$f(P_1) = P_1 \bigcup P_2 \bigcup P_3$$

$$f(P_2) = P_3$$

$$f(P_3) = P_3 \bigcup P_4$$

$$f(P_4) = P_1 \bigcup P_2 \bigcup P_3 \bigcup P_4$$

Markov!

$\Rightarrow \overrightarrow{s}, \; s \in \mathcal{A}$ Good coding

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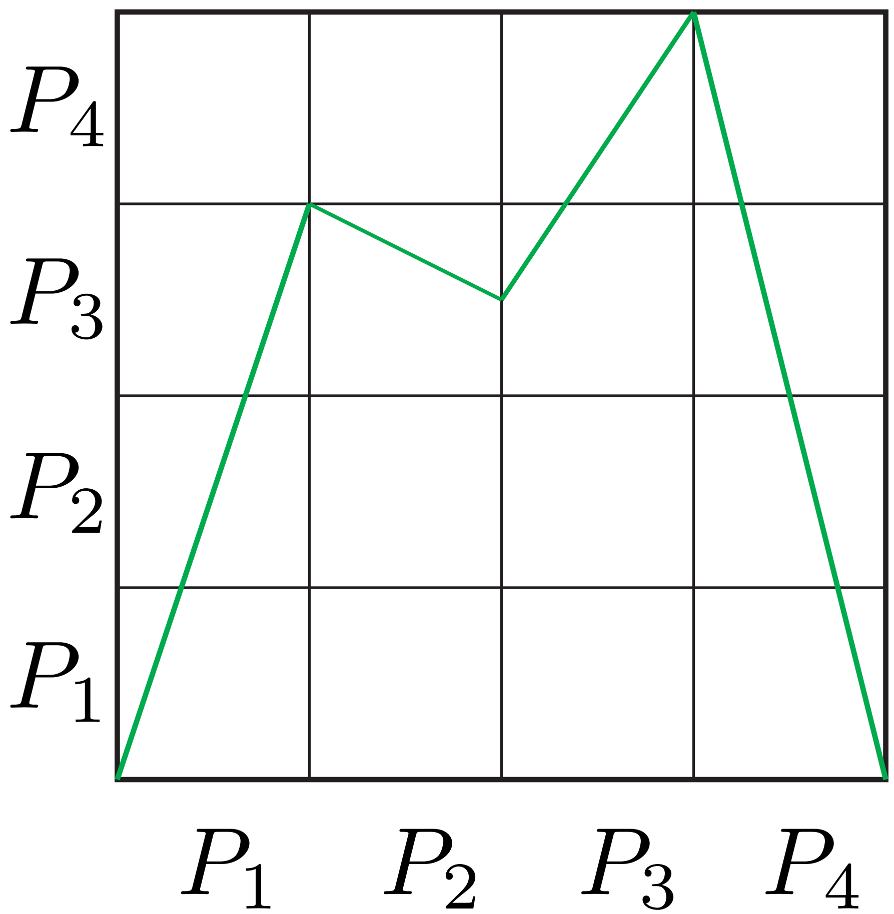
Markov Partitions for 1D Maps ...



$$f(P_2) \subset P_3$$

$$f(P_2) \neq \bigcup_i P_i$$

$$f(P_3) \subset P_3 \bigcup P_4$$

$$f(P_3) \neq \bigcup_i P_i$$

Not Markov!

$\Rightarrow \overrightarrow{s}, \;\; s \in \mathcal{A}$ Bad coding

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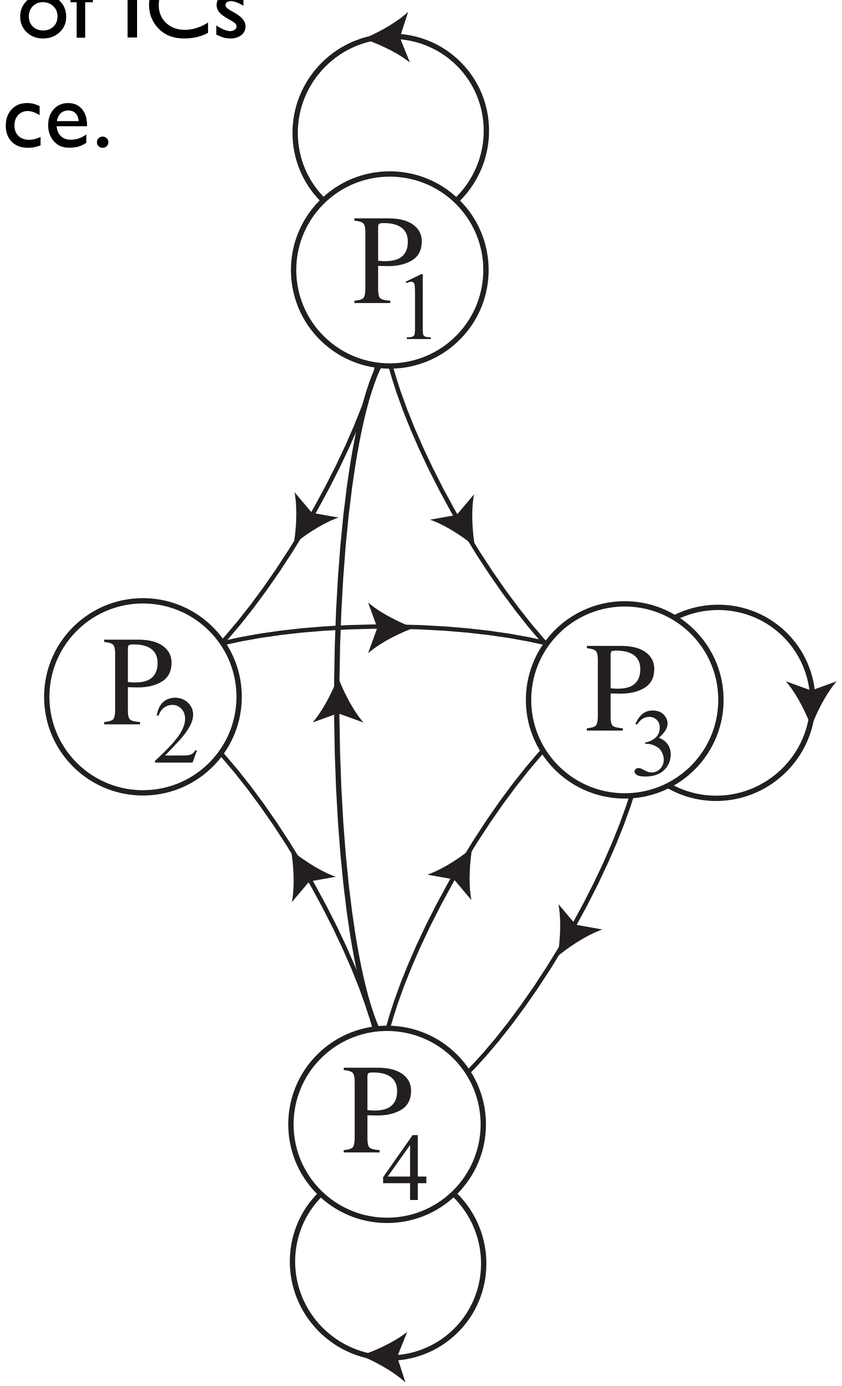
Why Markov Partition?

Symbol sequences track orbits:

Longer the sequence, the smaller the set of ICs that could have generated that sequence.

$$\lim_{L \to \infty} ||\Delta(\mathbf{s}^L)|| \to 0$$



Markov Partition is stronger:

Summarize map with a Markov chain over the partition elements.

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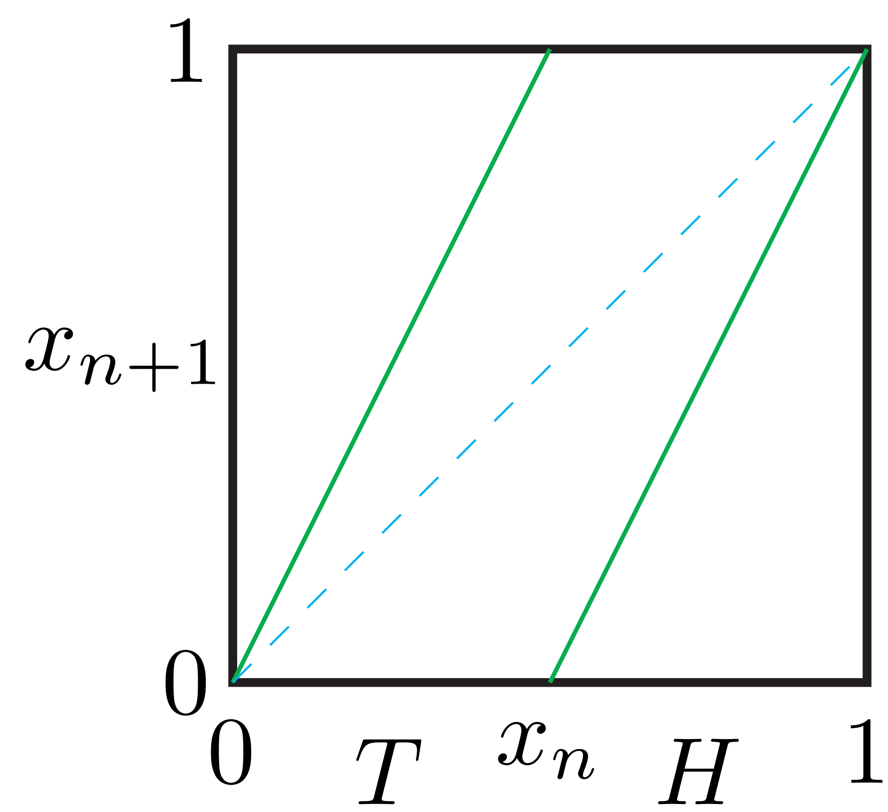
Markov partition for Shift map:

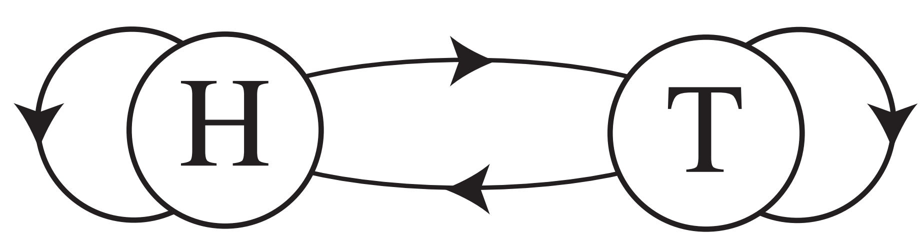


$$\mathcal{P} = \{T \sim (0, \tfrac{1}{2}), H \sim (\tfrac{1}{2}, 1)\}$$

$$f(P_T) = P_T \bigcup P_H \text{ \& } f|_{P_T} \text{ is monotone}$$

$$f(P_H) = P_T \bigcup P_H \text{ \& } f|_{P_H} \text{ is monotone}$$

Associated (topological) Markov chain:



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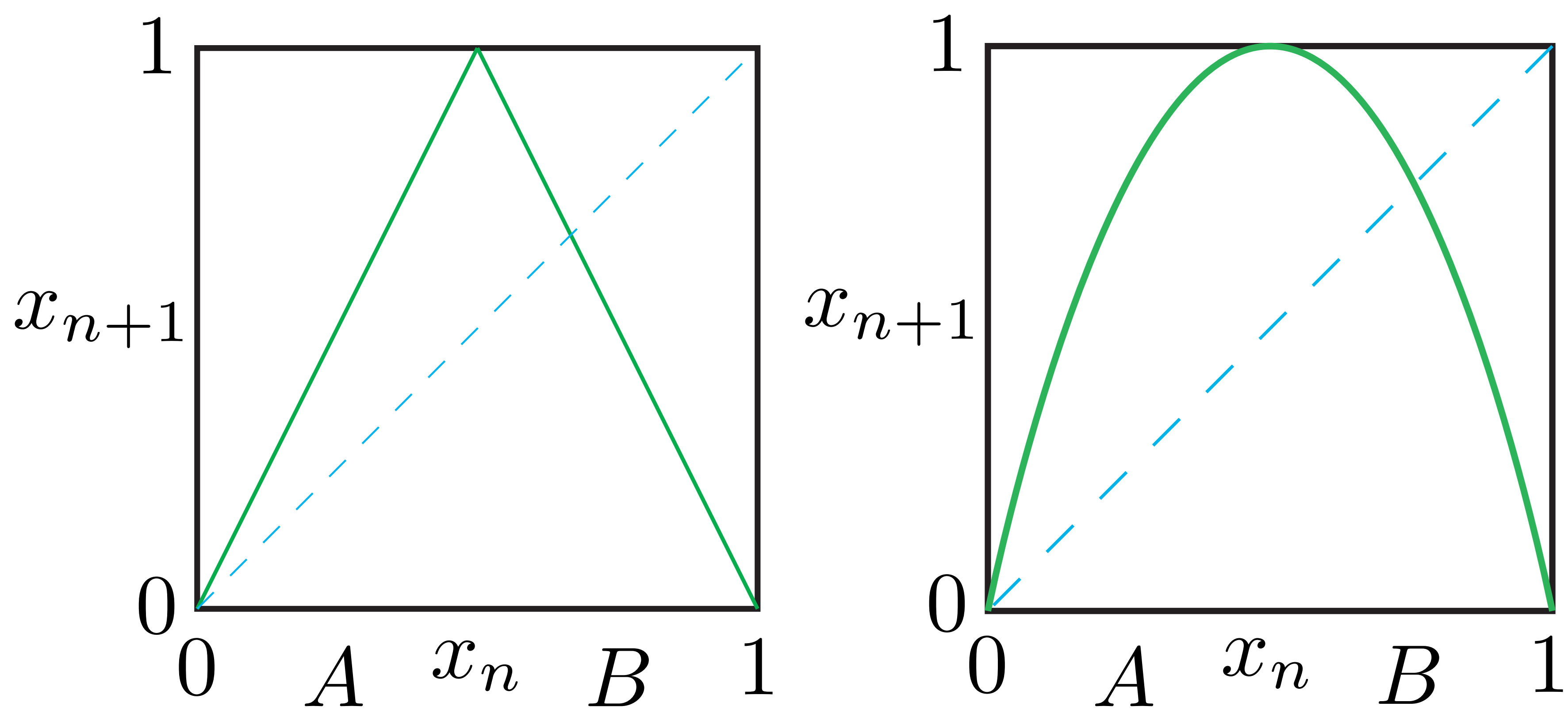
Markov partition for Tent and Logistic maps (Two-onto-One):

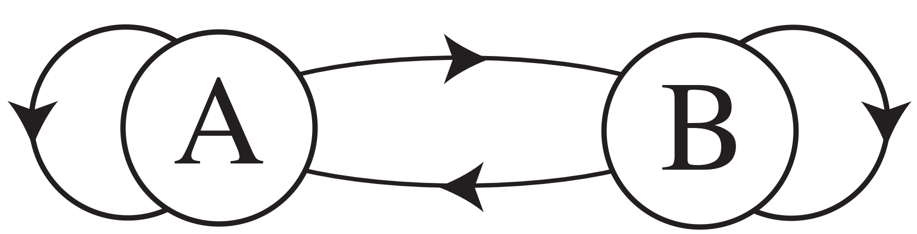
$$\mathcal{P} = \{A \sim (0, \tfrac{1}{2}), B \sim (\tfrac{1}{2}, 1)\}$$



$$f(P_A) = P_A \bigcup P_B \ \& \ f|_{P_A} \text{ is monotone}$$

$$f(P_B) = P_A \bigcup P_B \ \& \ f|_{P_B} \text{ is monotone}$$

Associated Markov chain:



Notice what is thrown away

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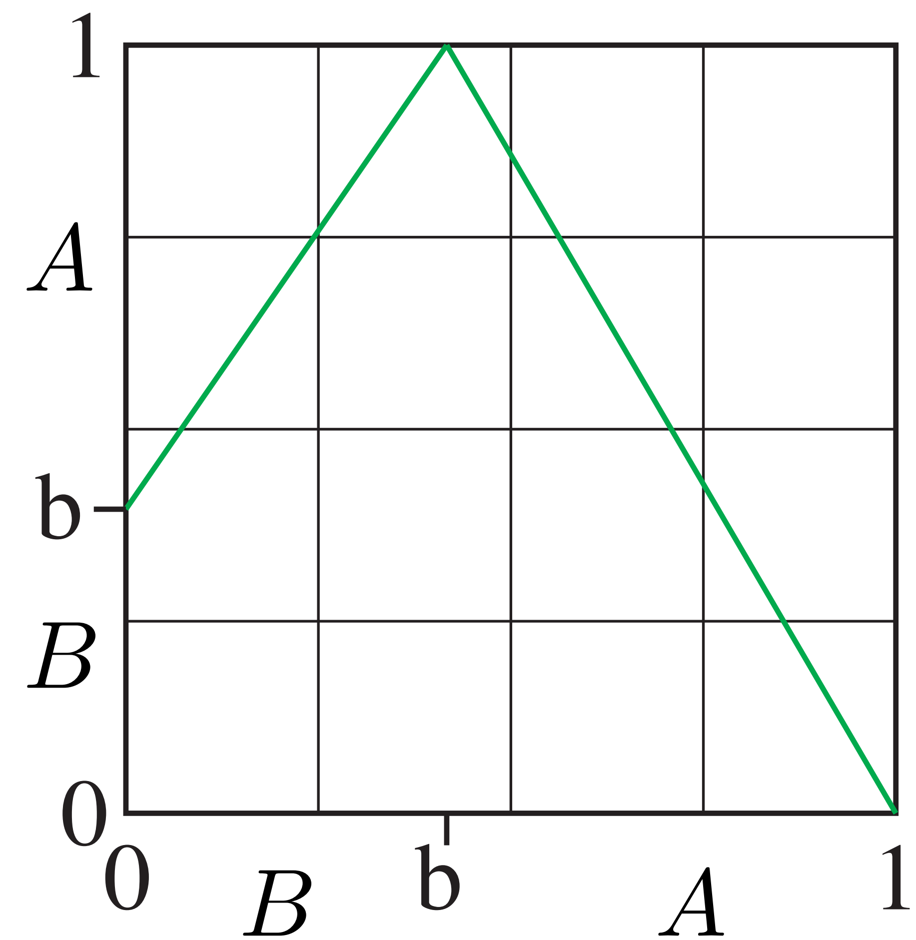
Measurement Theory ...

Markov partition for Golden Mean map:

$$x_{n+1} = \begin{cases} \phi x_n + b & 0 \leq x_n \leq b \\ (x_n - 1)/(b-1) & b < x_n \leq 1 \end{cases}$$

$$\phi = \frac{1+\sqrt{5}}{2} \qquad b = \frac{1}{1+\phi}$$

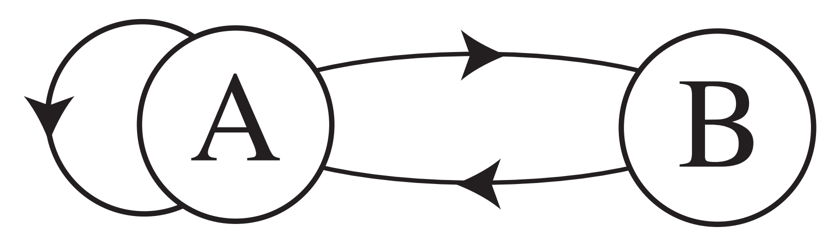


$$\mathcal{P} = \{B \sim (0,b), A \sim (b,1)\}$$

$$f(P_B) = P_A \;\&\; f|_{P_B} \text{ is monotone}$$

$$f(P_A) = P_B \bigcup P_A \;\&\; f|_{P_A} \text{ is monotone}$$

Markov chain is Golden Mean Process:



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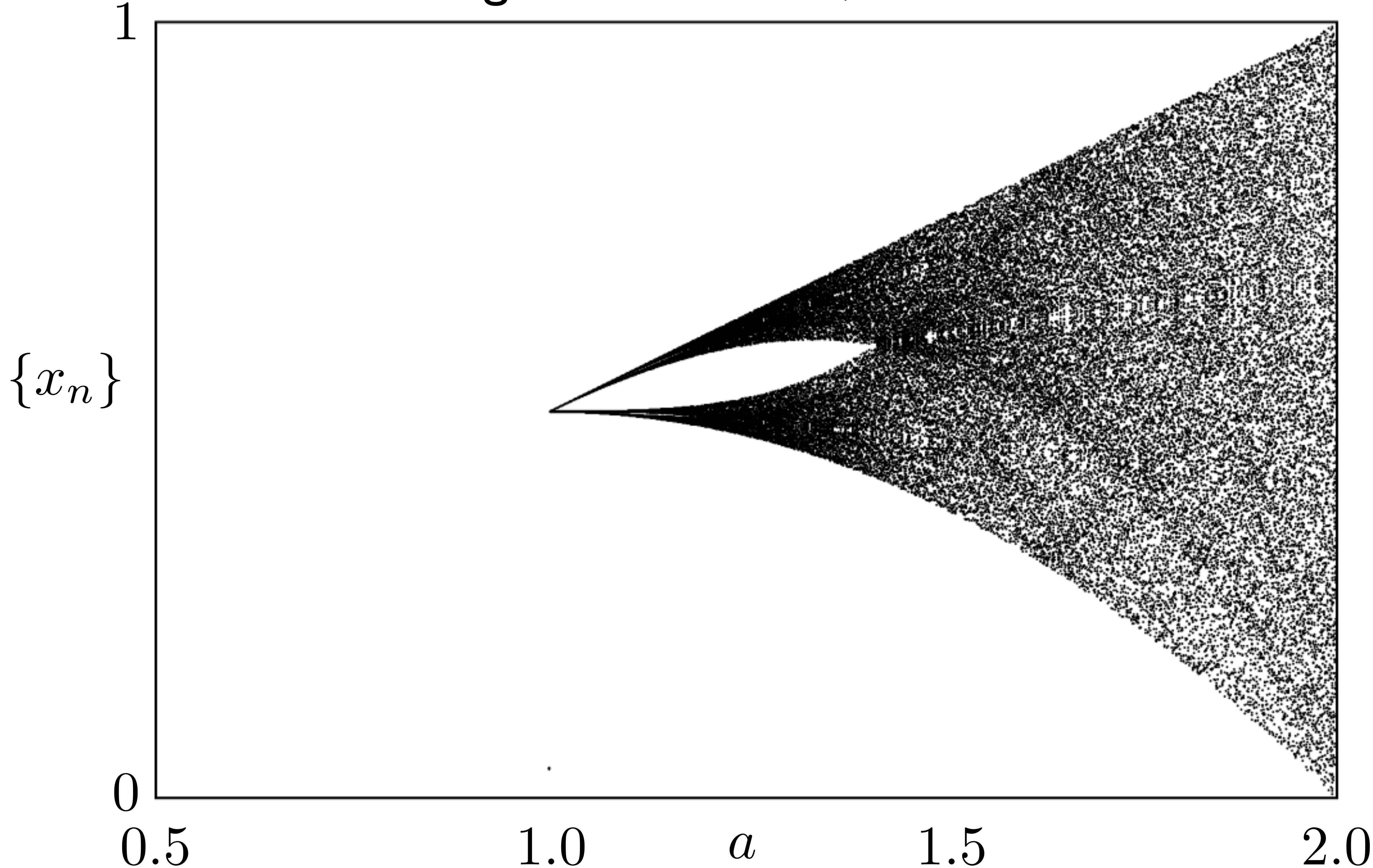
Measurement Theory ...

Markov partition for Tent map:

2 bands merge to 1 at $a = \sqrt{2}$.



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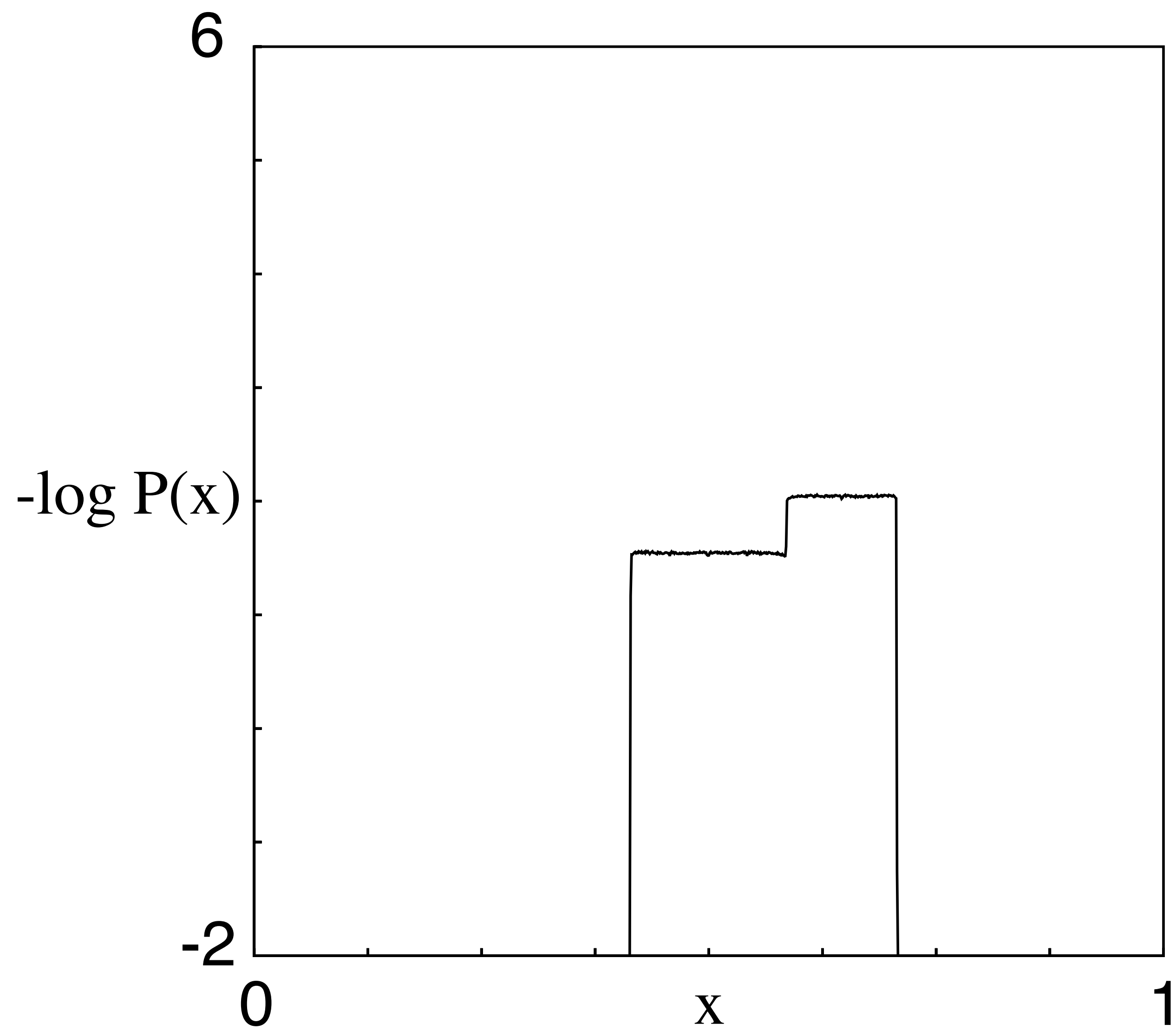


From Determinism to Stochasticity ...

Measurement Theory ...

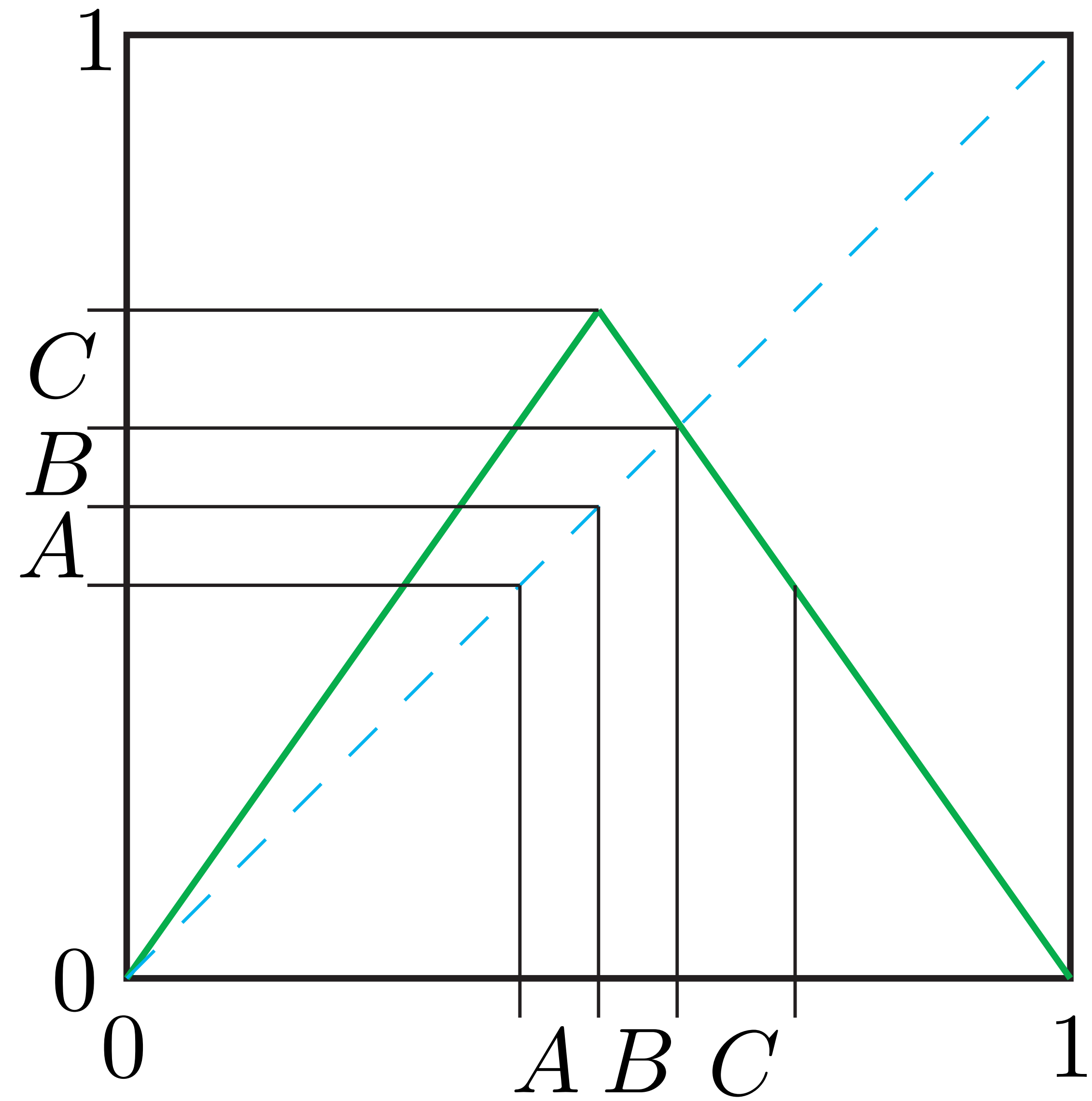
Markov partition for Tent map 2 bands merge to 1: $a = \sqrt{2}$

Attractor:

$$\Lambda = (f^2(\tfrac{1}{2}), f(\tfrac{1}{2}))$$

Partition:

$$\mathcal{P} = \{A \sim (f^2(\tfrac{1}{2}), \tfrac{1}{2}), \quad B \sim (\tfrac{1}{2}, f^3(\tfrac{1}{2})), \quad C \sim (f^3(\tfrac{1}{2}), f(\tfrac{1}{2}))\}$$



From Determinism to Stochasticity ...

Measurement Theory ...

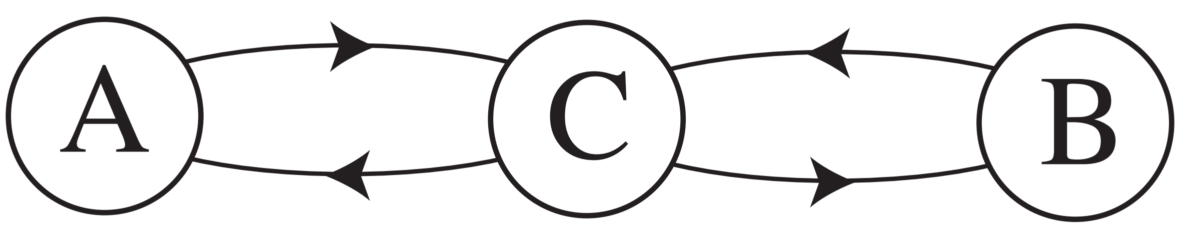
Markov partition for Tent map 2 bands merge to 1: $a = \sqrt{2}$

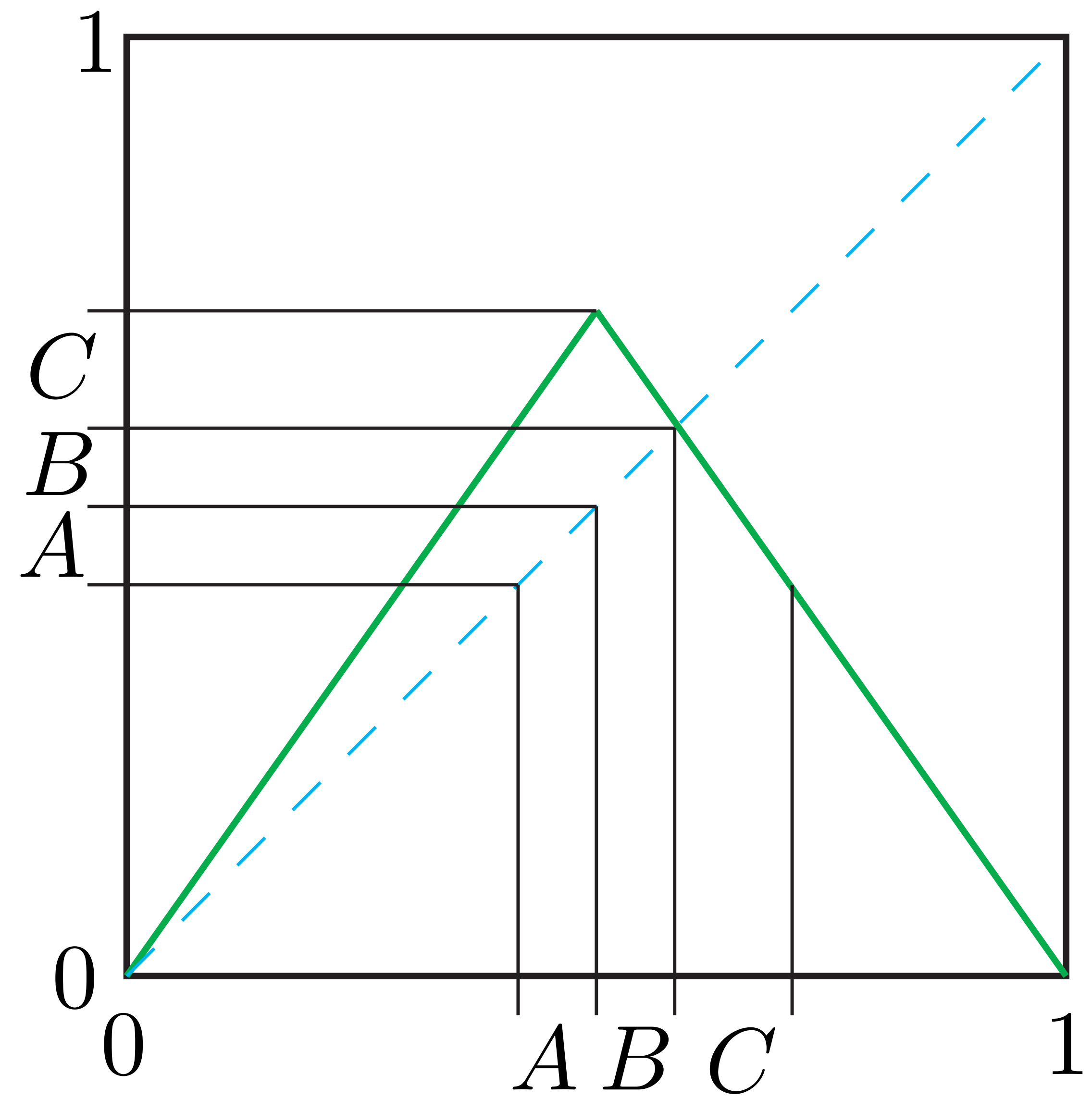
$$f(P_C) = P_A \bigcup P_B \ \& \ f|_{P_C} \text{ is monotone}$$

$$f(P_B) = P_C \ \& \ f|_{P_B} \text{ is monotone}$$

$$f(P_A) = P_C \ \& \ f|_{P_A} \text{ is monotone}$$

Markov chain:





From Determinism to Stochasticity ...

Reading for next lecture:

Lecture Notes.