From Determinism to Stochasticity
Stochastic Processes

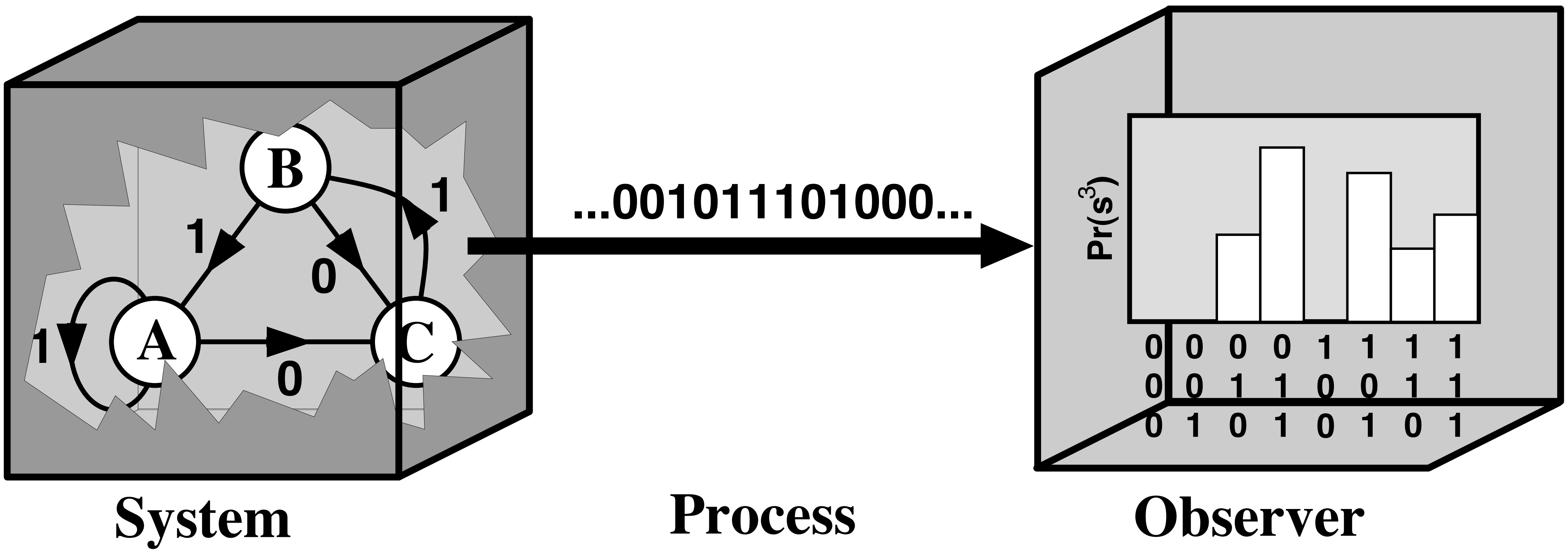
Reading for this lecture:

(These) *Lecture Notes*.

Note: We will skip the z-Transform and so the last slides here and also Computational Mechanics Reader (CMR) articles ZT and RI.

From Determinism to Stochasticity ...

The Measurement Channel:



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Stochastic Processes:

Chain of random variables:

$$\overleftrightarrow{S} \equiv \ldots S_{-2} S_{-1} S_0 S_1 S_2 \ldots$$

Random variable: S_t

Alphabet: $\mathcal{A} = \{1, 2, \ldots, k\}$

Realization:

$$\cdots s_{-2} s_{-1} s_0 s_1 s_2 \cdots \; ; \; s_t \in \mathcal{A}$$

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Stochastic Processes:

Chain of random variables: $\overleftrightarrow{S} = \overleftarrow{S}_t \overrightarrow{S}_t$

Past: $\overleftarrow{S}_t = \ldots S_{t-3} S_{t-2} S_{t-1}$

Future: $\overrightarrow{S}_t = S_t S_{t+1} S_{t+2} \ldots$

L-Block: $S_t^L \equiv S_t S_{t+1} \ldots S_{t+L-1}$

Word: $s_t^L \equiv s_t s_{t+1} \ldots s_{t+L-1} \in \mathcal{A}^L$

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Stochastic Processes ...

Process:

$$\Pr(\overleftrightarrow{S}) = \Pr(\ldots S_{-2} S_{-1} S_0 S_1 S_2 \ldots)$$

Sequence (or word) distributions:

$$\{\Pr(S_t^L) = \Pr(S_t S_{t+1} \ldots S_{t+L-1}) : S_t \in \mathcal{A}\}$$

Process:

$$\{\Pr(S_t^L) : \forall t, L\}$$

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Stochastic Processes ...

Word: $s_t^L = s_t s_{t+1} \ldots s_{t+L-1}$

Allowed (admissable) word: $\Pr(s_t^L) > 0$

Word distribution consistency conditions:

$$\Pr(s_t^{L-1}) \geq \Pr(s_t^L)$$

$$\Pr(s_t^{L-1}) = \sum_{\{s_{t+L-1}\}} \Pr(s_t^L)$$

$$\Pr(s_t^{L-1}) = \sum_{\{s_t\}} \Pr(s_t^L)$$

Subword closed: All subwords in s_t^L are admissable.

Processes are subword closed.

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Types of Stochastic Process:

Stationary process:

$$\Pr(S_t S_{t+1} \ldots S_{t+L-1}) = \Pr(S_0 S_1 \ldots S_{L-1})$$

Ignore process's starting condition.
Or, over many realizations.
$\Pr(\cdot)$ is independent of time.

Assume stationarity, unless otherwise stated.

Drop time indices:

$$S_t^L \to S^L$$

$$s_t^L \to s^L$$

Types of Stochastic Process ...

Uniform Process:

Equal-length sequences occur with same probability

$$U^L : \ \Pr(s^L) \ = \ 1/|\mathcal{A}|^L$$

Example: Fair coin

$$\mathcal{A} = \{H, T\}$$

$$\Pr(H) = \Pr(T) = 1/2$$

$$\Pr(s^L) = 2^{-L}$$

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Types of Stochastic Process ...

Independent, Identically Distributed (IID) Process:

$$\Pr(\overleftrightarrow{S}) = \ldots \Pr(S_t)\Pr(S_{t+1})\Pr(S_{t+2})\ldots$$

$$\Pr(S_t) = \Pr(S_\tau), \ \forall \ t, \tau$$

Example: Biased coin

$$\Pr(H) = p$$

$$\Pr(T) = 1 - p = q$$

$$\Pr(s^L) = p^n q^{L-n}$$

Number of heads in sequence: n

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Types of Stochastic Process ...

R-Block Process:

$$\Pr(\overleftrightarrow{S}) = \cdots \Pr(S_1 \ldots S_R) \Pr(S_{R+1} \ldots S_{2R}) \cdots$$

Example: A 2-block process with no consecutive 0s

$$\mathcal{A} = \{0, 1\}$$

$$\Pr(00) = 0$$

$$\Pr(01) = 0$$

$$\Pr(10) = \tfrac{1}{2}$$

$$\Pr(11) = \tfrac{1}{2}$$

Noisy Period-2 Process

$$\Pr(111010) = \Pr(11)\Pr(10)\Pr(10)$$

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Types of Stochastic Process ...

Markov Process:

$$\Pr(\overleftrightarrow{S}) = \ldots \Pr(S_{t+1}|S_t)\Pr(S_{t+2}|S_{t+1})\Pr(S_{t+3}|S_{t+2})\ldots$$

Example: No Consecutive 0s (Golden Mean Process)

$$\mathcal{A} = \{0, 1\}$$

$$\Pr(0|0) = 0$$

$$\Pr(1|0) = 1$$

$$\Pr(0|1) = 1/2$$

$$\Pr(1|1) = 1/2$$

Not Noisy Period-2 Process: GMP @ L = 4 has 0110.

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Types of Stochastic Process ...

Order-R Markov Process:

$$\Pr(S_i | \ldots, S_{i-2}, S_{i-1}) = \Pr(S_i | S_{i-R}, \ldots, S_{i-1})$$

Order-R processes are more general than R-block processes:

$$\begin{aligned}
\Pr(S_1 S_2 S_3 S_4) &= \Pr(S_1) \Pr(S_2 | S_1) \Pr(S_3 | S_2) \Pr(S_4 | S_3) \\
&= \Pr(S_1 S_2) \frac{\Pr(S_2 S_3)}{\Pr(S_2)} \frac{\Pr(S_3 S_4)}{\Pr(S_3)} \\
&= \Pr(S_1 S_2) \Pr(S_3 S_4)
\end{aligned}$$

Only when blocks are independent: $\dfrac{\Pr(S_2 S_3)}{\Pr(S_2) \Pr(S_3)} = 1$

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Types of Stochastic Process ...

Hidden Markov Process:

Internal Order-R Markov Process: $\Pr(\overleftrightarrow{S})$

$$\Pr(S_t | \ldots S_{t-2} S_{t-1}) = \Pr(S_t | S_{t-R} \ldots S_{t-1})$$

$$s_t \in \mathcal{A}$$

Observed via a function of the internal sequences

$$\overleftrightarrow{Y} = f(\overleftrightarrow{S})$$

Measurement alphabet: $y_t \in \mathcal{B}$

Measurement random variables: $\overleftrightarrow{Y} = \ldots Y_{-2} Y_{-1} Y_0 Y_1 \ldots$

Observation process: $\Pr(\overleftrightarrow{Y} \mid \overleftrightarrow{S})$

Observed process: $\Pr(\overleftrightarrow{Y})$

Block Distribution: $\Pr(Y^L)$

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Types of Stochastic Process ...

Hidden Markov Process ...

Example: The Even Process

Internal Process: Golden Mean

$$s_t \in \{0, 1\}$$

Observation Process: $y_t \in \{a, b\}$

$$Y_t = f(S_{t-1} S_t)$$

$$y_t = \begin{cases} a, & s_{t-1} s_t = 11 \\ b, & s_{t-1} s_t = 01 \text{ or } 10 \end{cases}$$

$$\overleftrightarrow{s} = \texttt{110111011110101111011} \ldots$$

$$\overleftrightarrow{y} = \texttt{.\ abbaabbaaabbbbaaaabba} \ldots$$

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Models of Stochastic Processes:

Markov chain model of a Markov process:

States: $v \in \mathcal{A} = \{1, \ldots, k\}$

$$\overleftrightarrow{V} = \ldots V_{-2} V_{-1} V_0 V_1 \ldots$$

Transition matrix: $T_{ij} = \Pr(v_{t+1} | v_t) \equiv p_{vv'}$

$$T = \begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{pmatrix}$$

Stochastic matrix: $\displaystyle\sum_{i=1}^{k} T_{ij} = 1$

Exercise:

An R-block Markov process is a Markov chain with $k = |\mathcal{A}|^R$

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Models of Stochastic Processes ...

Markov chain ...

State distribution:

$$\vec{p}_V = (\Pr(v=1), \Pr(v=2), \ldots, \Pr(v=k))$$

$$\vec{p}_V = (p_1, p_2, \ldots, p_k)$$

Evolve probability distribution:

$$\vec{p}_n = \vec{p}_{n-1} T$$

Initial distribution: $\vec{p}_0$

$$\vec{p}_n = \vec{p}_0 T^n$$

State sequence distribution:

Path: $v^L = v_0 v_1 v_2 \ldots v_{L-1}$

$$\Pr(v^L) = p(v_0) p(v_1|v_0) p(v_2|v_1) \ldots p(v_{L-1}|v_{L-2})$$

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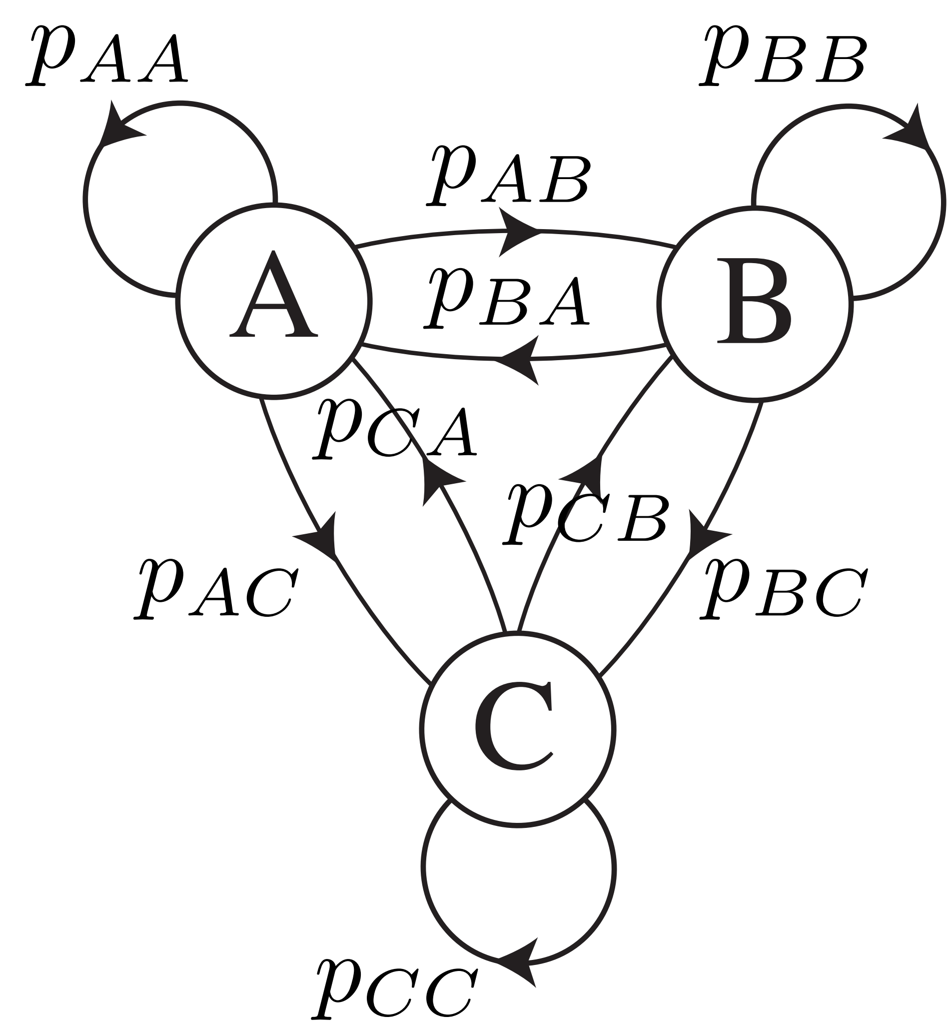
Models of Stochastic Processes ...

Markov chain ...

Example: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}$$

$$p_{AA} + p_{AB} + p_{AC} = 1$$
$$p_{BA} + p_{BB} + p_{BC} = 1$$
$$p_{CA} + p_{CB} + p_{CC} = 1$$



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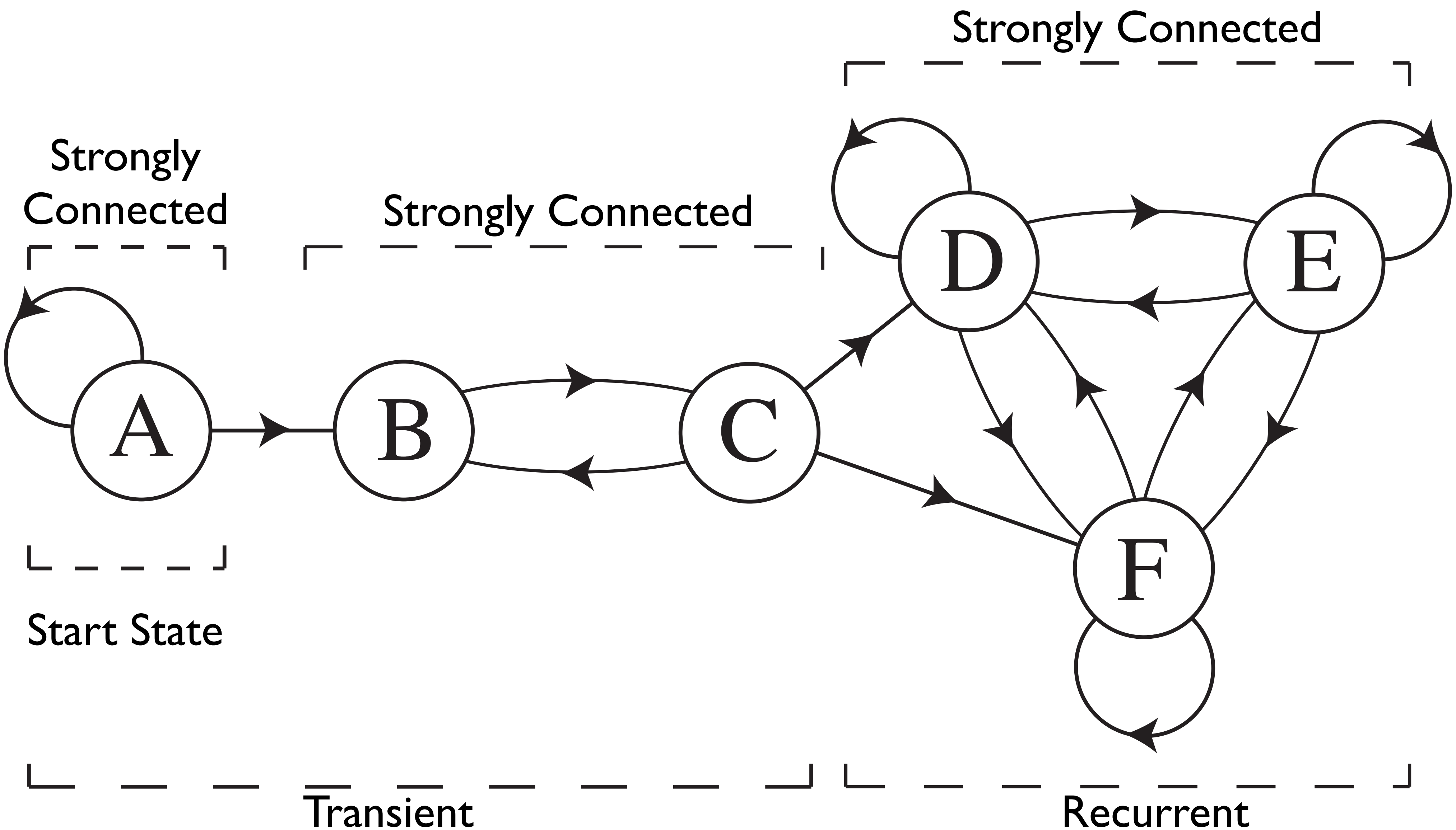
Models of Stochastic Processes ...

Kinds of state:



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Statistical equilibrium:
$$\pi = \lim_{n\to\infty} \vec{p}_n$$
$$= \vec{p}_0 \lim_{n\to\infty} T^n$$

Principal (left) eigenvector: $\pi = \pi\, T$ (Eigenvalue = 1)

Normalized in probability: $$\sum_{i=1}^{k} \pi = 1$$

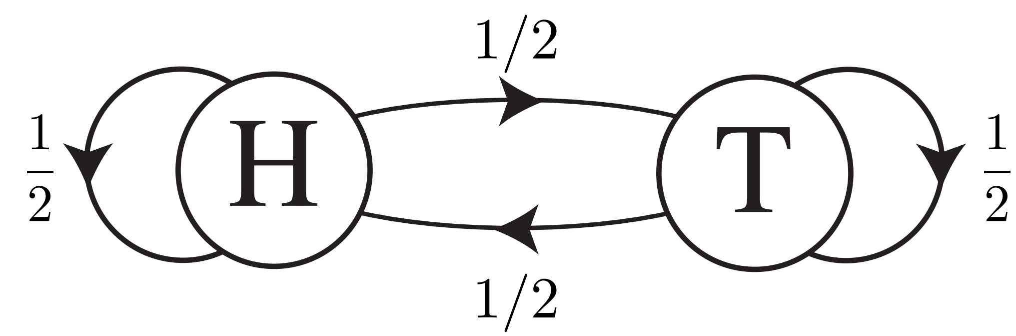
Asymptotic state sequence distribution:
$$v^L = v_0 v_1 v_2 \ldots v_{L-1}$$
$$\Pr(v^L) = \pi(v_0) p(v_1|v_0) p(v_2|v_1) \cdots p(v_{L-1}|v_{L-2})$$

Models of Stochastic Processes ...

Example:

Fair Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



$$\pi = (1/2, 1/2)$$

$$\Pr(H) = \Pr(T) = 1/2$$

General uniform process: Markov chain has as many states as symbols, with uniform transition probabilities leaving them going to all states.

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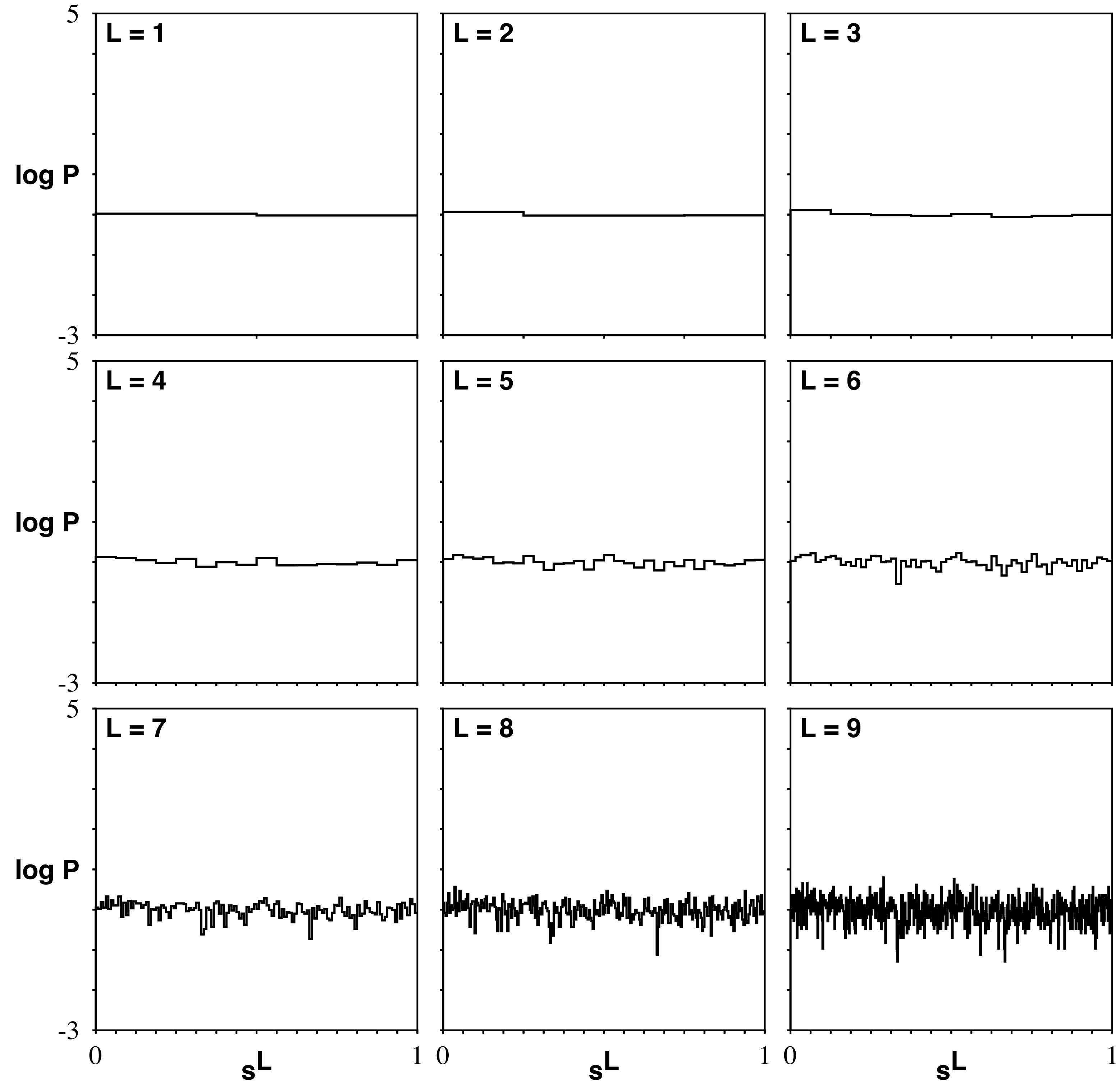
Models of Stochastic Processes ...

Example:

Fair Coin ...

Sequence Distribution: $\Pr(v^L) = 2^{-L}$



Word as binary fraction:

$$s^L = s_1 s_2 \ldots s_L$$

$$\text{“}s^L\text{”} = \sum_{i=1}^{L} \frac{s_i}{2^i}$$

$$s^L \in [0, 1]$$

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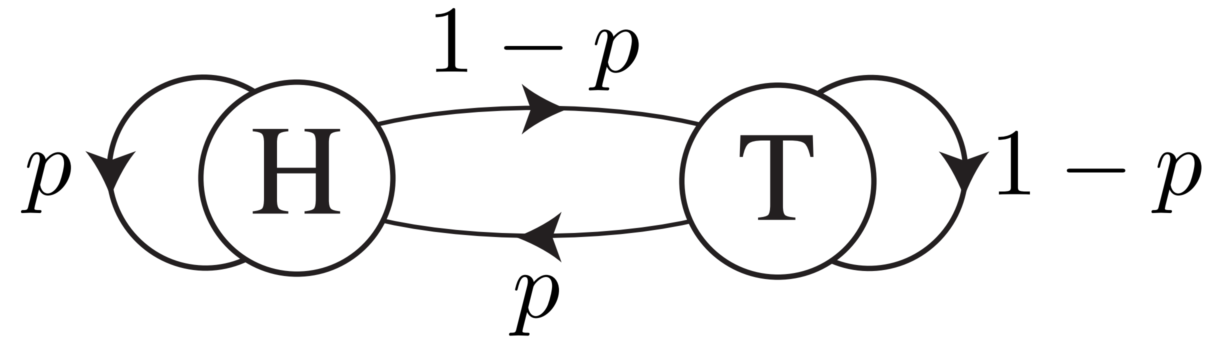
Models of Stochastic Processes ...

Example:

Biased Coin: $\mathcal{A} = \{H, T\}$

$$T = \begin{pmatrix} p & 1-p \\ p & 1-p \end{pmatrix}$$



$$\pi = (p, 1-p)$$

IID processes: Markov chain has as many states as symbols. Transitions leave each state and go to all states. Transitions entering state i have the same probability, which is π_i.

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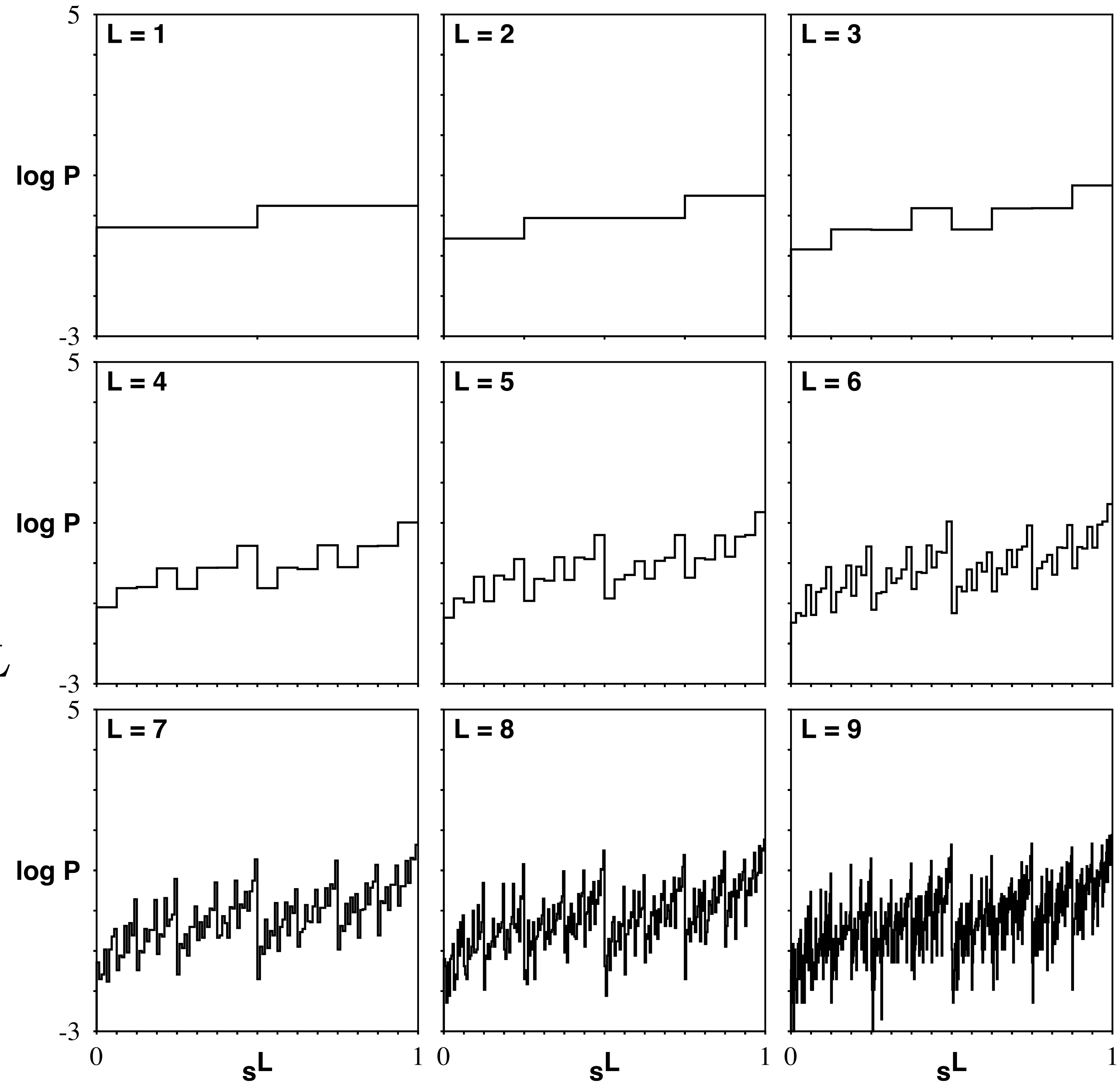
Models of Stochastic Processes ...

Example:

Biased Coin ...



Sequence Distribution:

$$\Pr(s^L) = p^n (1-p)^{L-n},$$

$$n = \text{Number } H\text{s in } s^L$$

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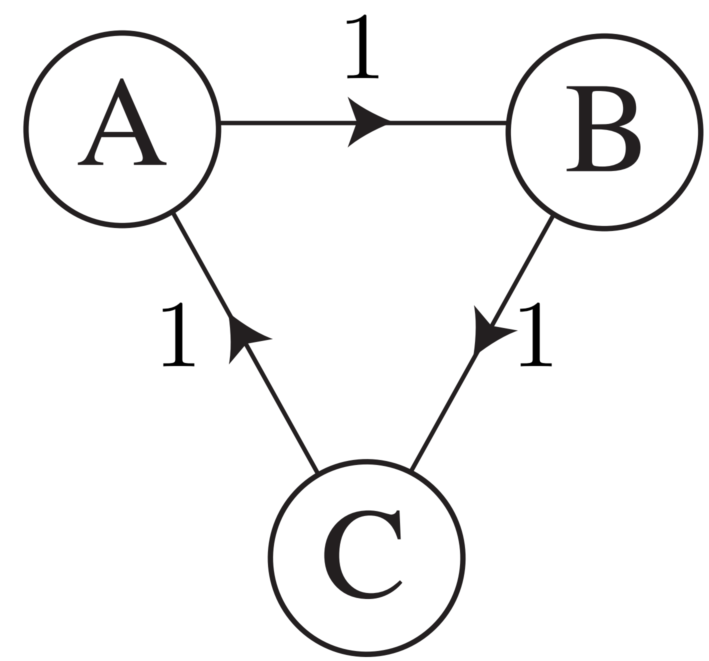
Example:

Periodic: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$



$$\pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$ Careful!

Sequence distribution:

$$\Pr(A) = \Pr(B) = \Pr(C) = \tfrac{1}{3}$$

$$\Pr(AB) = \Pr(BC) = \Pr(CA) = \tfrac{1}{3} \qquad \Pr(s^2) = 0 \quad \text{otherwise}$$

$$\Pr(ABC) = \Pr(BCA) = \Pr(CAB) = \tfrac{1}{3} \quad \Pr(s^3) = 0 \quad \text{otherwise}$$

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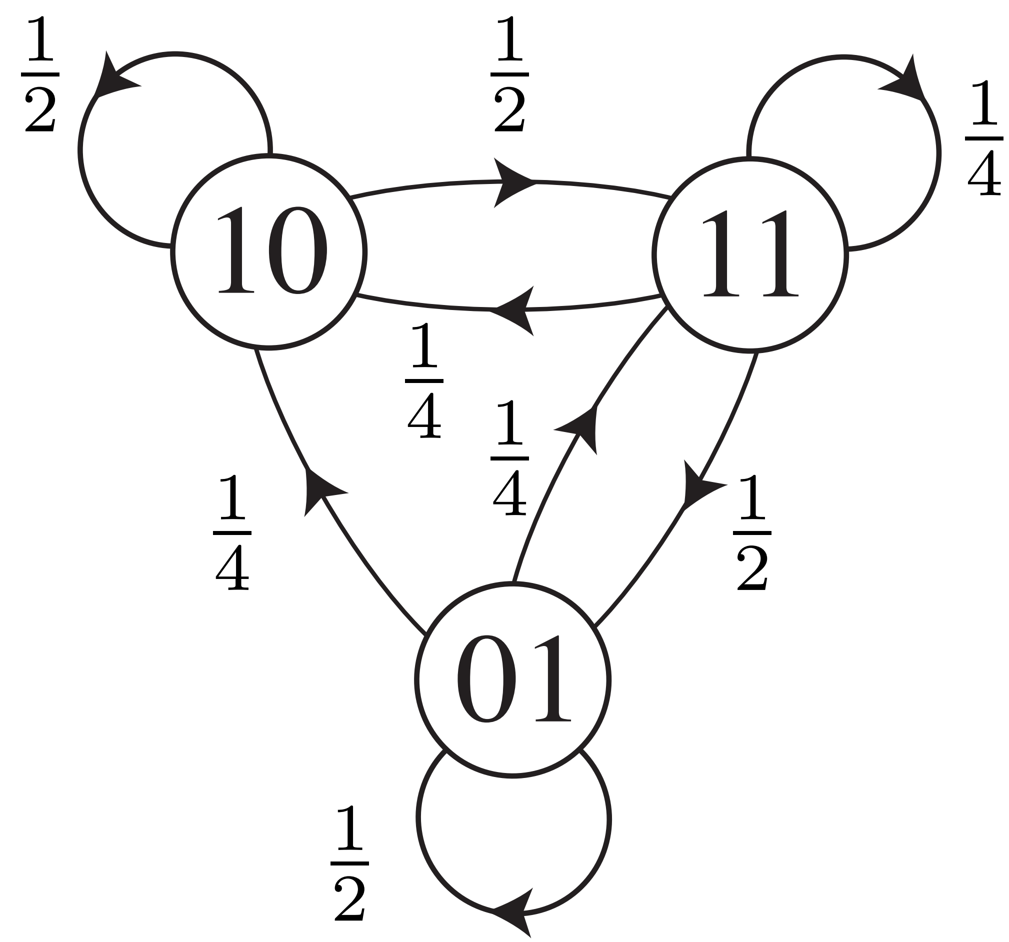
Models of Stochastic Processes ...

Example:

Golden Mean over 2-Blocks: $\mathcal{A} = \{10, 01, 11\}$

$$T = \begin{array}{c} \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} 10 & 01 & 11 \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \end{array}$$

$$\pi = \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right)$$



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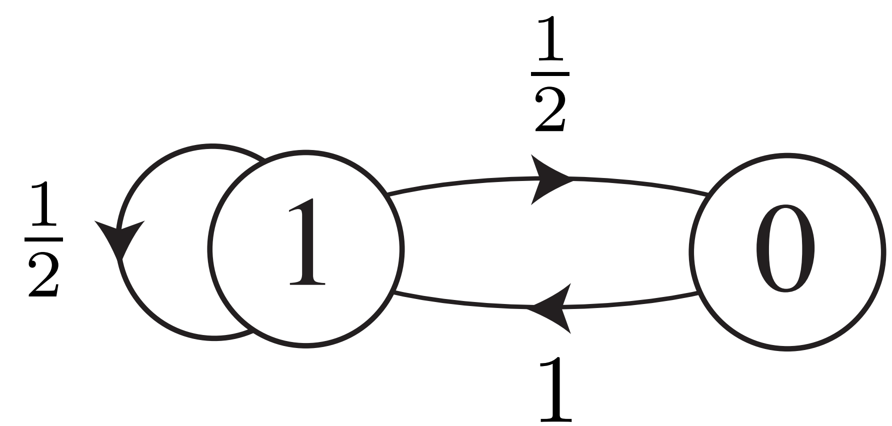
Example ...

Golden Mean over 1-Blocks: $\mathcal{A} = \{0, 1\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$\pi = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$



Also an order-1 Markov chain. Minimal order.

Previous model and this:

Different presentations of the Golden Mean Process

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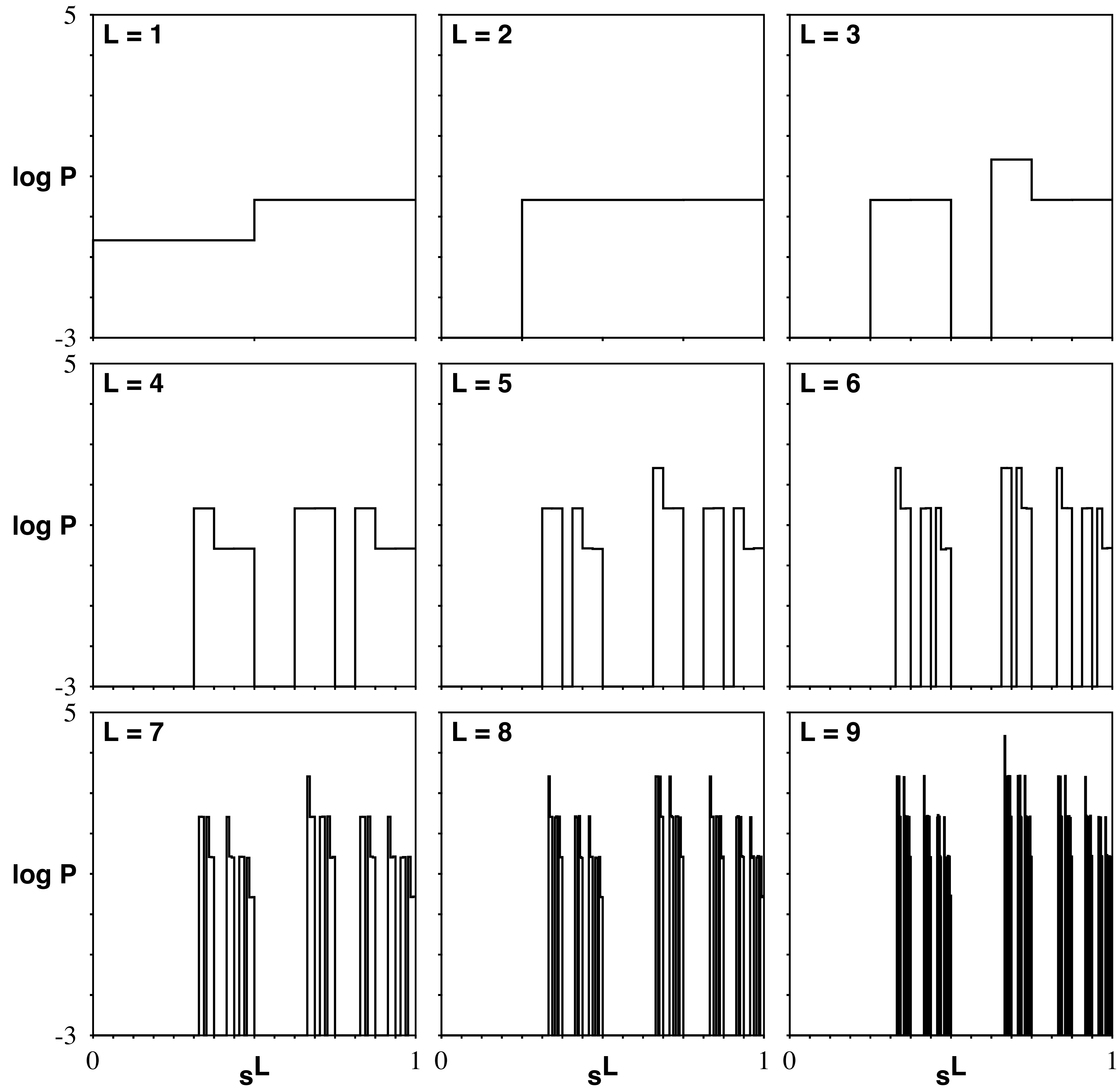
Models of Stochastic Processes ...

Example:

Golden mean:



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Models of Stochastic Processes ...

Two Lessons:

Structure in the behavior: $\mathrm{supp}\ \Pr(s^L)$

Structure in the distribution of behaviors: $\Pr(s^L)$

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Models of Stochastic Processes ...

Hidden Markov Models of Processes:

Internal states: $v \in \mathcal{A}$

Transition matrix: $T = \Pr(v'|v), \; v, v' \in \mathcal{A}$

Observation: Symbol-labeled transition matrices

$$T^{(s)} = \Pr(v', s|v), \; s \in \mathcal{B}$$

$$T = \sum_{s \in \mathcal{B}} T^{(s)}$$

Stochastic matrices:

$$\sum_j T_{ij} = \sum_j \sum_s T_{ij}^{(s)} = 1$$

From Determinism to Stochasticity ...

Models of Stochastic Processes ...

Hidden Markov Models ...

Internal state distribution: $\vec{p}_V = (p_1, p_2, \ldots, p_k)$

Evolve internal distribution: $\vec{p}_n = \vec{p}_0 T^n$

State sequence distribution: $v^L = v_0 v_1 v_2 \ldots v_{L-1}$

$$\Pr(v^L) = \pi(v_0) p(v_1 | v_0) p(v_2 | v_1) \cdots p(v_{L-1} | v_{L-2})$$

Observed sequence distribution: $s^L = s_0 s_1 s_2 \ldots s_{L-1}$

$$\Pr(s^L) = \sum_{v^L \in \mathcal{A}^L} \pi(v_0) p(v_1, s_1 | v_0) p(v_2, s_2 | v_1) \cdots p(v_{L-1}, s_{L-1} | v_{L-2})$$

No longer 1-1 map between internal & observed sequences:

Multiple state sequences can produce *same* observed sequence.

From Determinism to Stochasticity ...

Models of Stochastic Processes ...

Hidden Markov Models ...

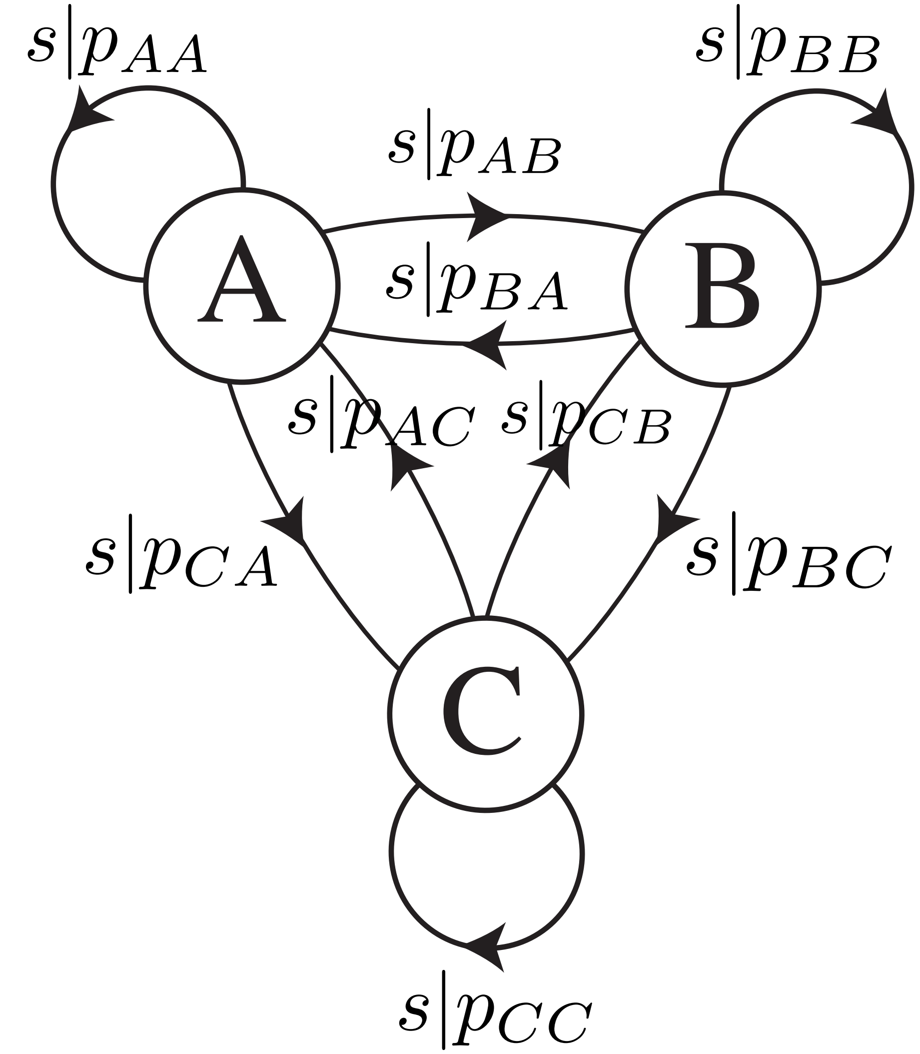
Internal: $\mathcal{A} = \{A, B, C\}$

$$T = \begin{pmatrix} p_{AA} & p_{AB} & p_{AC} \\ p_{BA} & p_{BB} & p_{BC} \\ p_{CA} & p_{CB} & p_{CC} \end{pmatrix}$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(s)} = \begin{pmatrix} p_{AA;s} & p_{AB;s} & p_{AC;s} \\ p_{BA;s} & p_{BB;s} & p_{BC;s} \\ p_{CA;s} & p_{CB;s} & p_{CC;s} \end{pmatrix}$$

$$p_{AA} = \sum_{s \in \mathcal{B}} p_{AA;s}$$



symbol | transition probability

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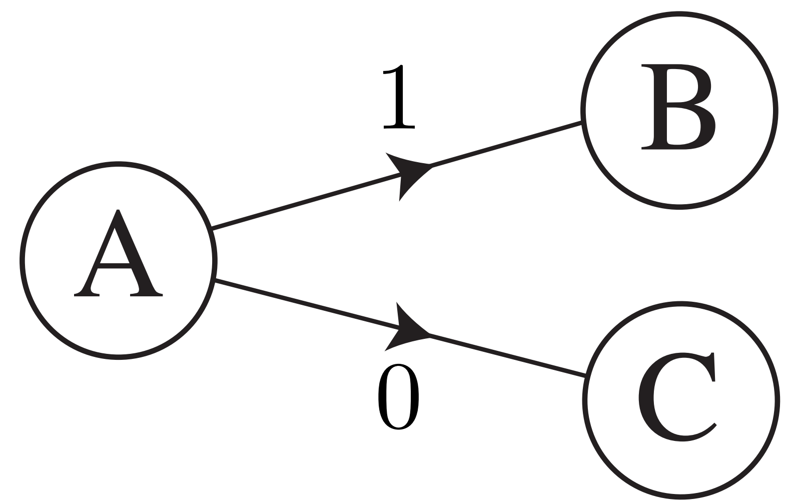
Models of Stochastic Processes ...

Types of Hidden Markov Model:

Edge (v, s, v'), its probability: $\Pr(v, s, v')$.

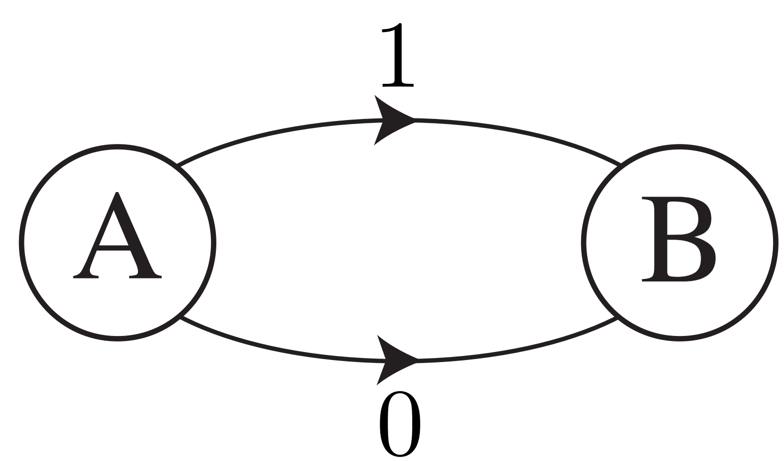
Unifilar HMM: State + symbol “determine” next state

$$\Pr(v'|v,s) = \begin{cases} 1 \\ 0 \end{cases}$$



$$\Pr(v', s|v) = p(s|v)$$

$$\Pr(v'|v) = \sum_{x \in \mathcal{A}} p(v', s|v)$$



(In automata theory: “Unifilar” = “deterministic”.)

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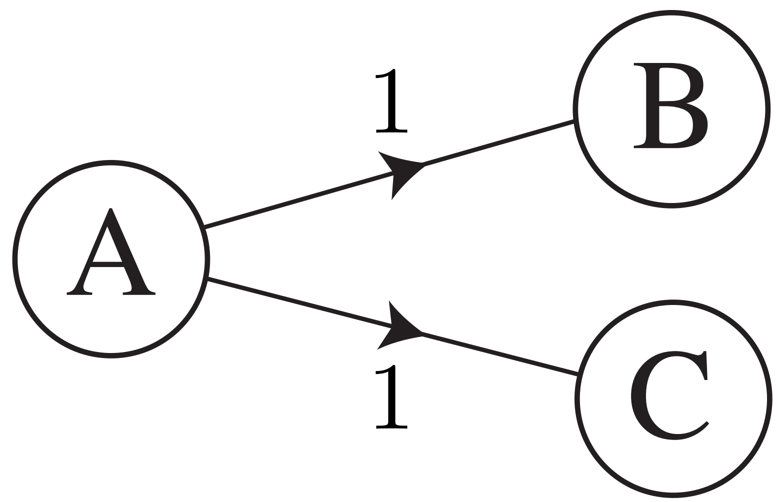
Models of Stochastic Processes ...

Types of Hidden Markov Model:

Nonunifilar HMM: At least one violation of unifilarity



Consequence:

Multiple internal edge paths can generate
same observed sequence.

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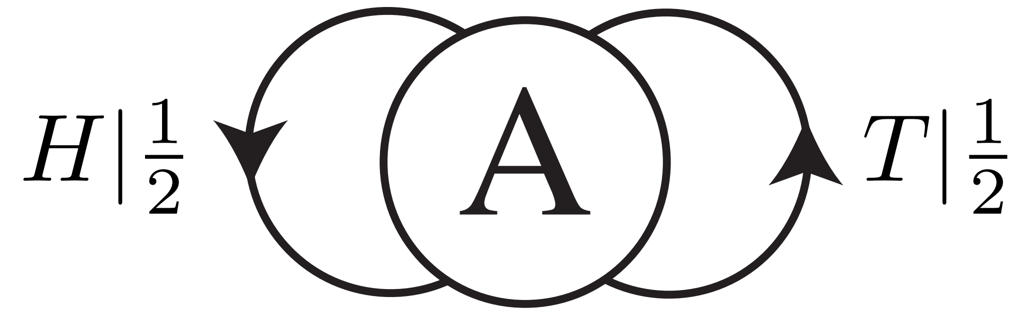
Models of Stochastic Processes ...

Example:

Fair Coin as a unifilar HMM:



Internal: $\mathcal{A} = \{A\}$

$$T = (1) \qquad \pi_V = (1)$$

Observed: $\mathcal{B} = \{H, T\}$

$$T^{(0)} = \left(\tfrac{1}{2}\right) \qquad T^{(1)} = \left(\tfrac{1}{2}\right)$$

One one state!

HMM for general uniform process has more transitions with equal transition probabilities, but still needs only one state.

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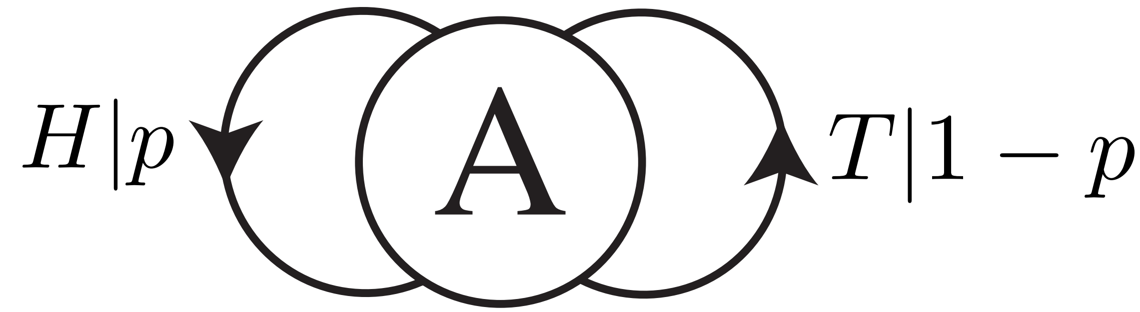
Models of Stochastic Processes ...

Example:

Biased Coin as a unifilar HMM:



Internal: $\mathcal{A} = \{A\}$

$$T = (1) \qquad \pi_V = (1)$$

Observed: $\mathcal{B} = \{H, T\}$

$$T^{(0)} = \left(\tfrac{1}{2}\right) \qquad T^{(1)} = \left(\tfrac{1}{2}\right)$$

Also, one one state!

HMMs for IID processes need have only one state!

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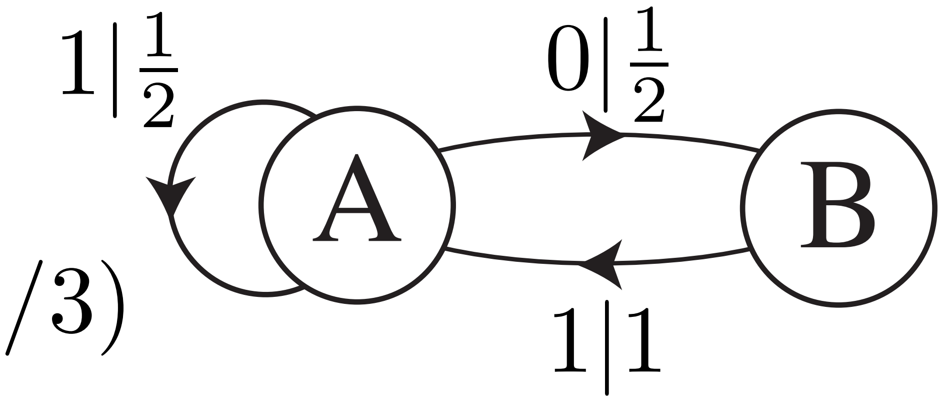
Models of Stochastic Processes ...

Example:

Golden Mean Process as a unifilar HMM:



Internal: $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{pmatrix}$$

Initial ambiguity only: At most 2-to-1 mapping

$$BA^{n-1} = 1^n$$

$$AA^{n-1} = 1^n$$

Sync'd: $s = 0 \Rightarrow v = B$

$s = 1 \Rightarrow v = A$

Irreducible forbidden words: $\mathcal{F} = \{00\}$

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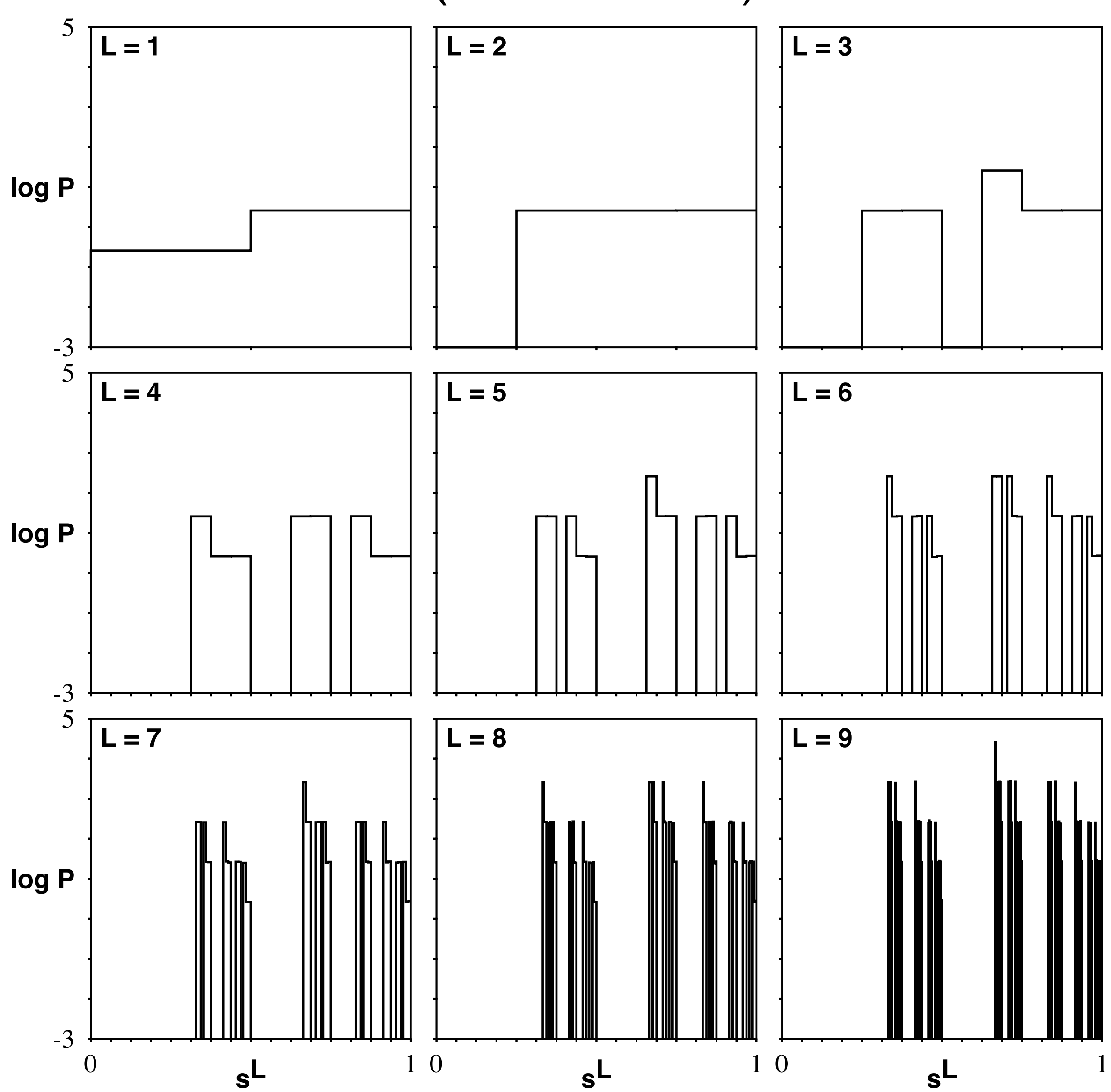
Models of Stochastic Processes ...

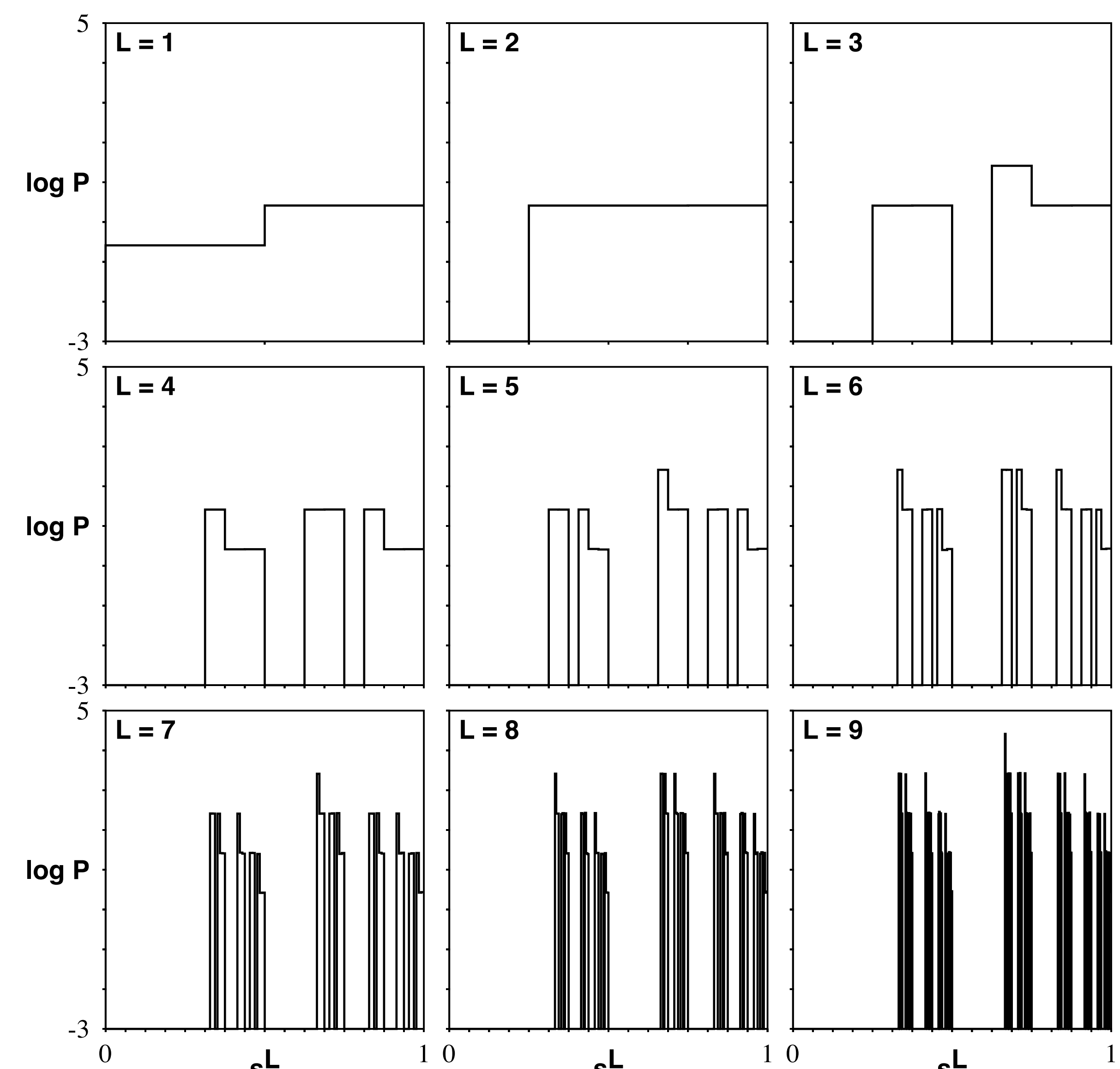
Example:

Golden Mean Process ... Sequence distributions:

Internal state sequences (A = 1; B = 0)



Observed sequences



Same!

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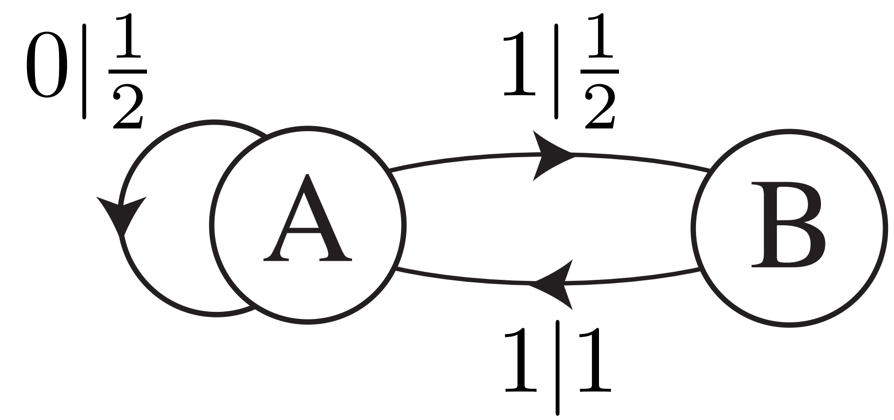
Models of Stochastic Processes ...

Example:

Even Process as a unifilar HMM:



Internal (= GMP): $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} \quad \pi_V = (2/3, 1/3)$$

Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

$$v^L = \ldots AABAABABAA \ldots$$

$$s^L = \ldots . \;\; 0\,1\,1\,0\,1\,1\,1\,1\,0 \ldots \qquad s^L = \{\ldots 01^{2n}0 \ldots\}$$

Irreducible forbidden words: $\mathcal{F} = \{010, 01110, 0111110, \ldots\}$

No finite-order Markov process can model the Even process!

Lesson: Finite Markov Chains are a subset of HMMs.

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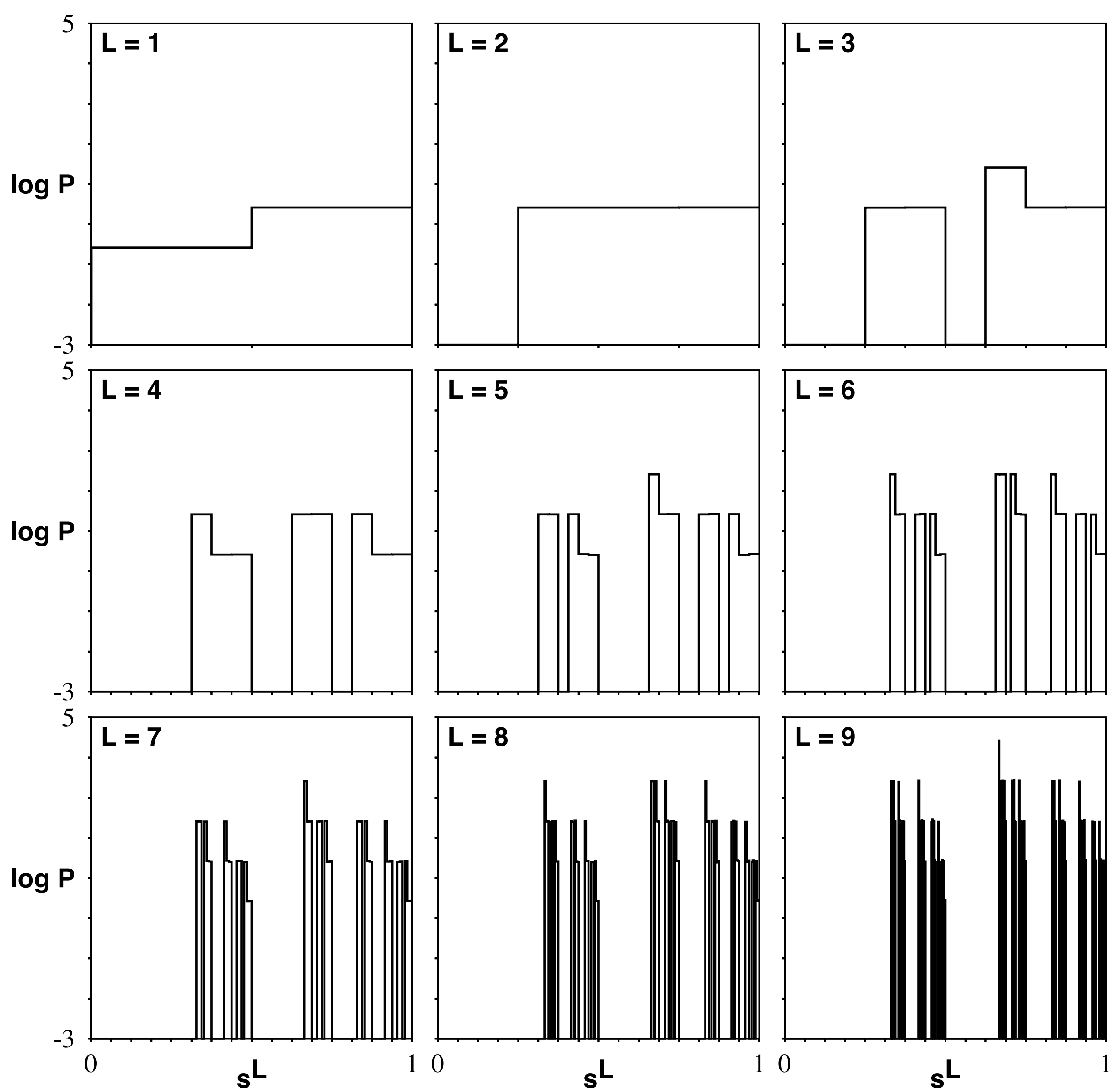
Models of Stochastic Processes ...

Example:

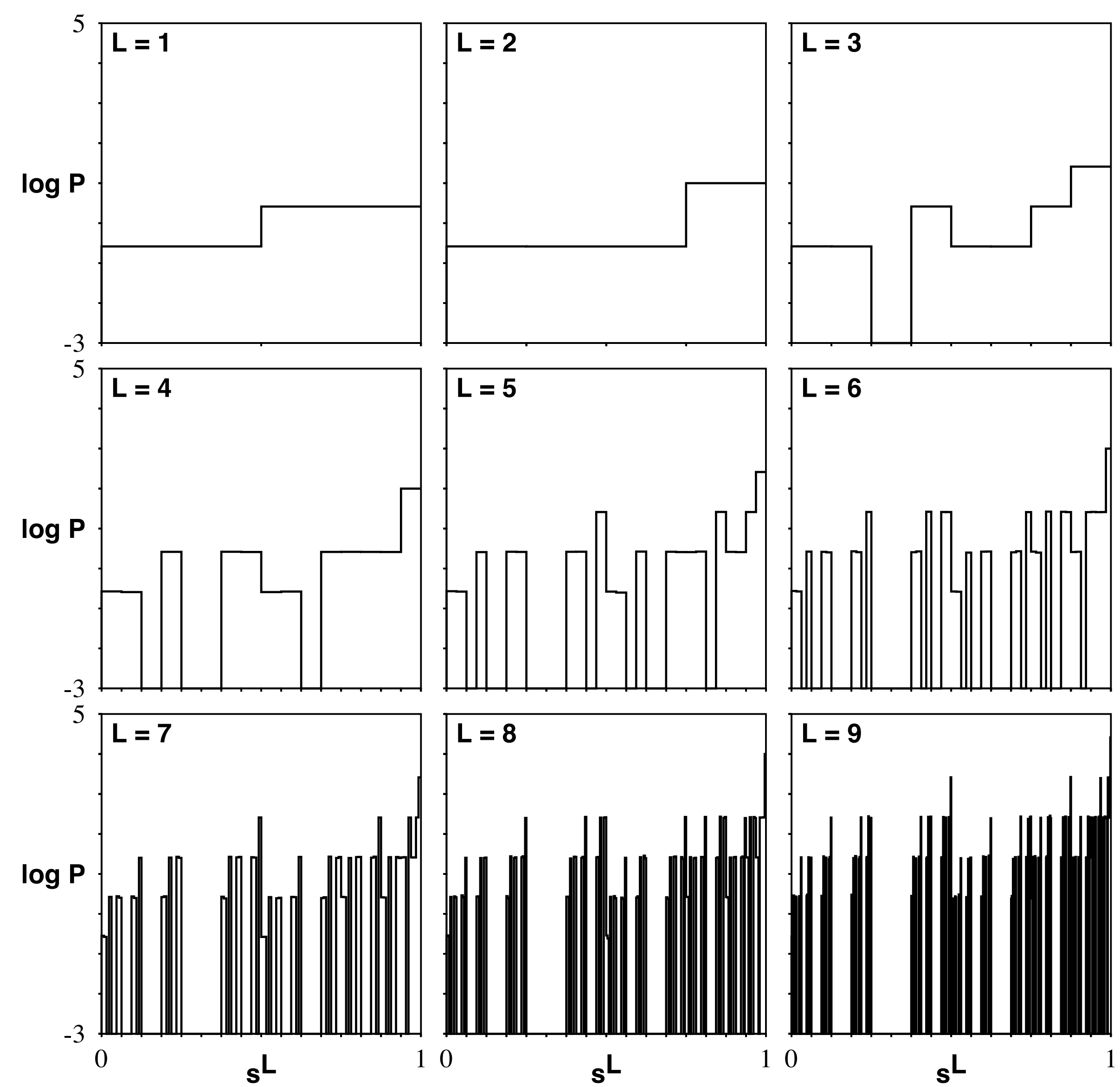
Even Process ... Sequence distributions:

Internal states (= GMP)

(A = 1; B = 0)



Observed sequences



Rather different!

From Determinism to Stochasticity ...

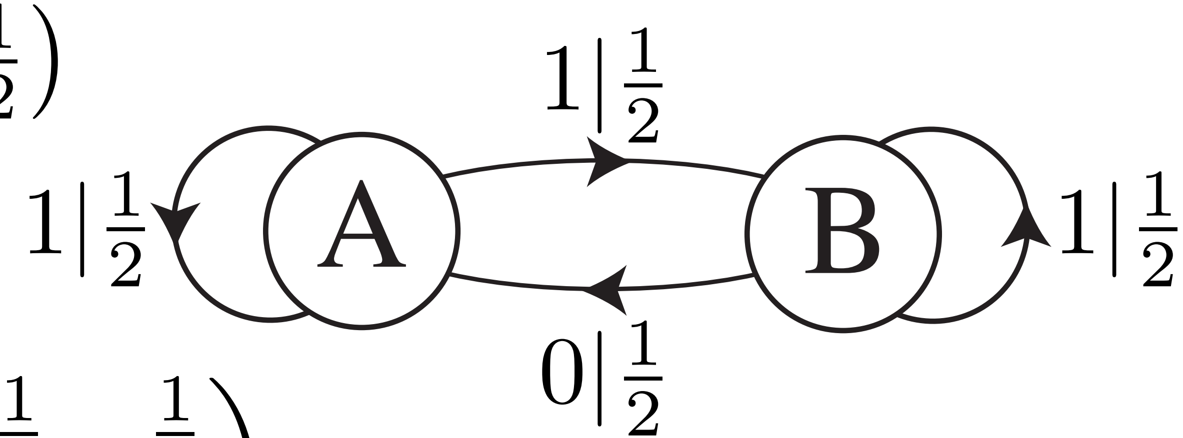
Models of Stochastic Processes ...

Example:

Simple Nonunifilar Source:

Internal (= Fair Coin): $\mathcal{A} = \{A, B\}$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \pi_V = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$



Observed: $\mathcal{B} = \{0, 1\}$

$$T^{(0)} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} \quad T^{(1)} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Many to one: $1111111 \Leftarrow \begin{cases} AAAAAAAA\ldots \\ ABBBBBBB\ldots \\ AABBBBBB\ldots \\ AAABBBBB\ldots \\ \ldots \\ BBBBBBBB\ldots \end{cases}$

Is there a unifilar HMM presentation of the observed process?

From Determinism to Stochasticity ...

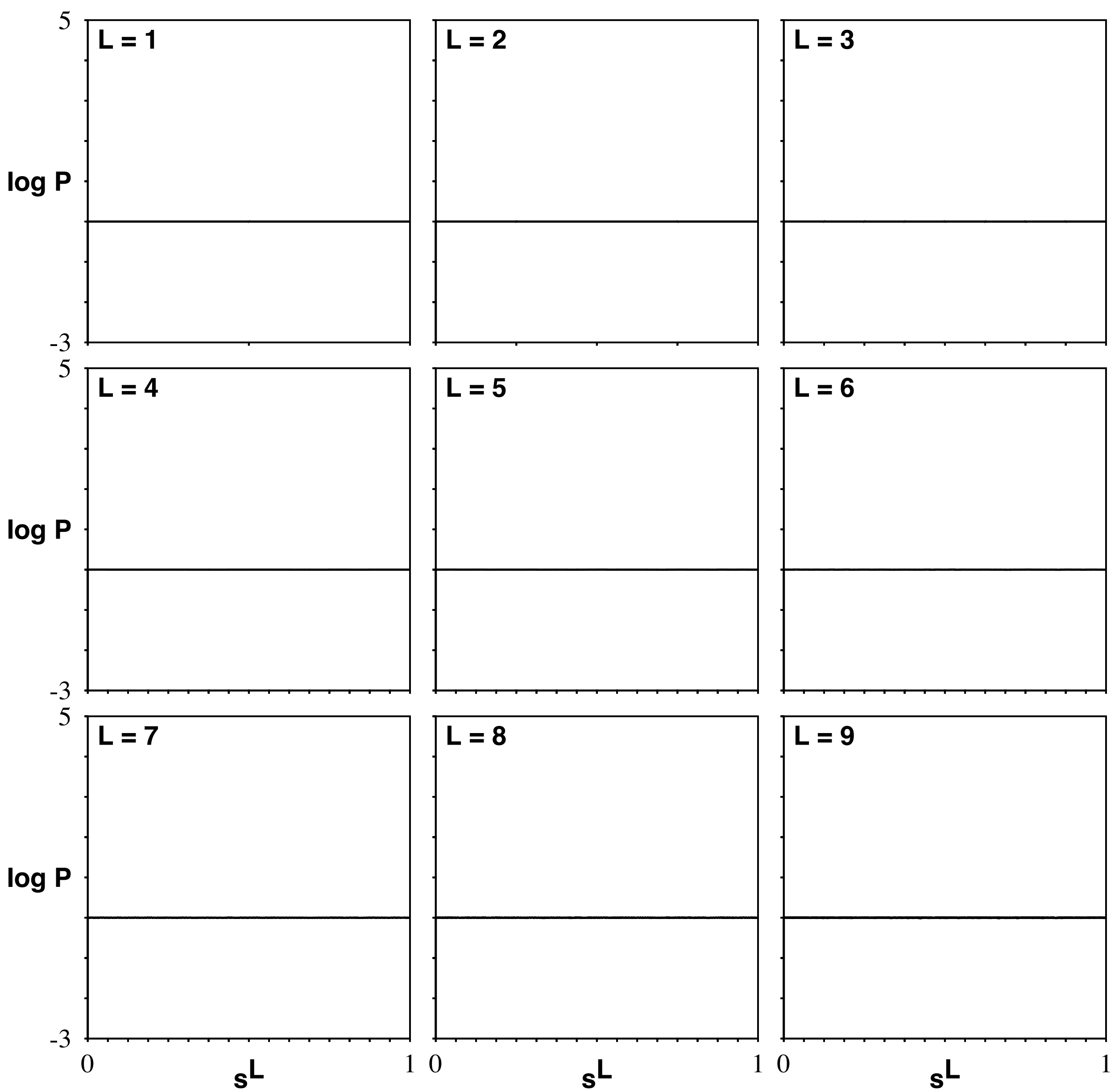
Models of Stochastic Processes ...

Example:

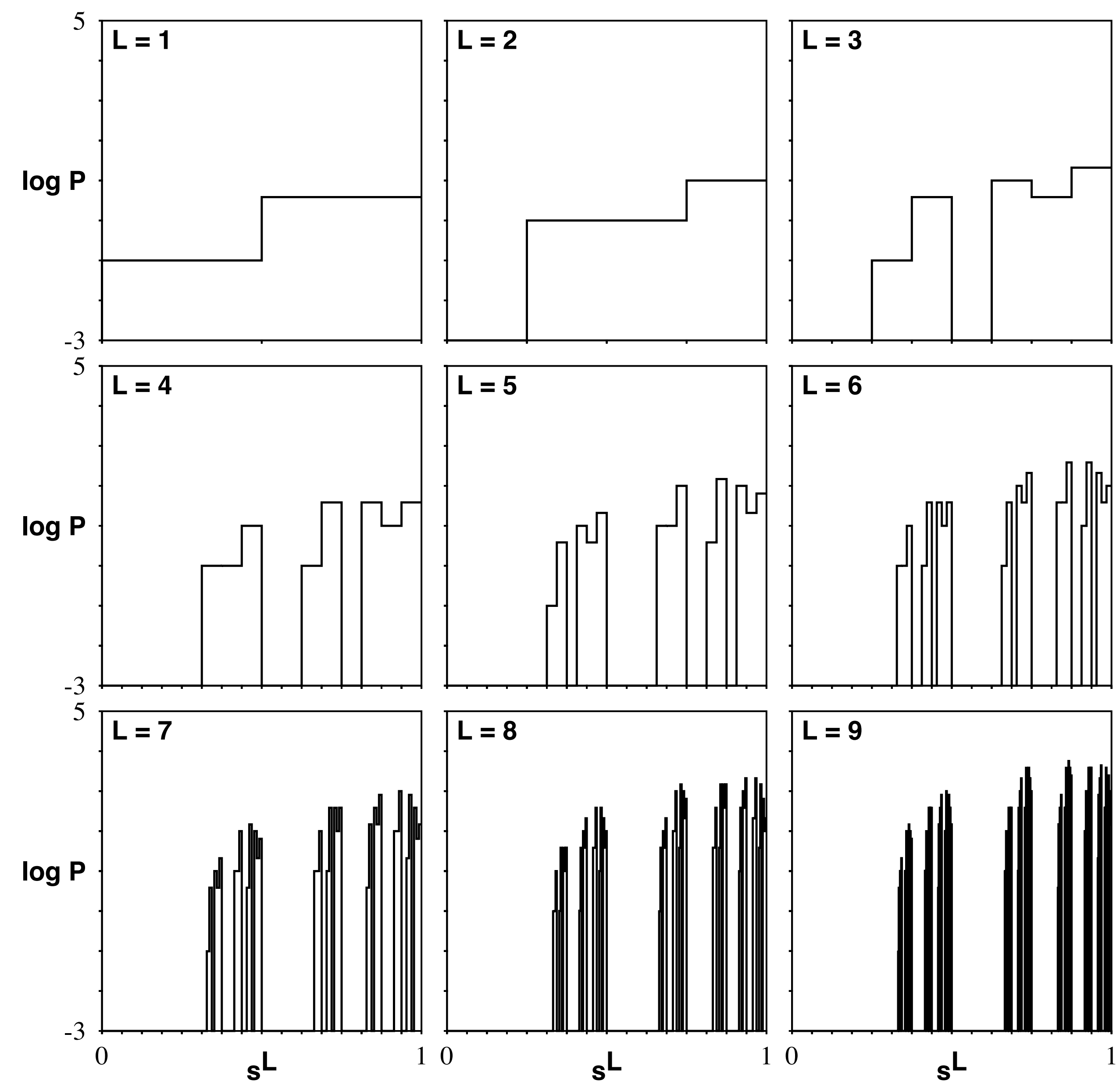
Simple Nonunifilar Process ...

Internal states (= Fair coin)

(A = 1; B = 0)

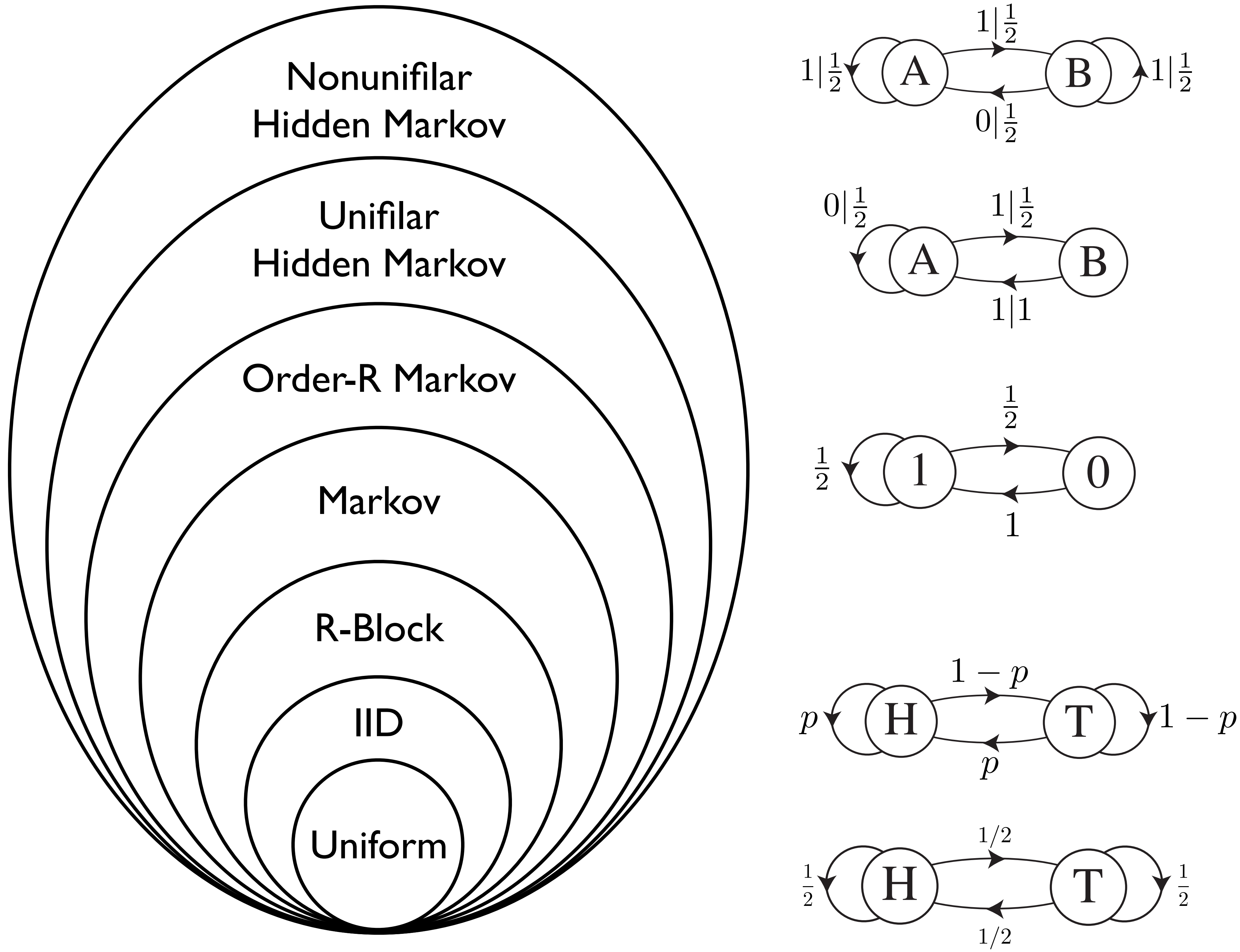


Observed sequences



From Determinism to Stochasticity ...

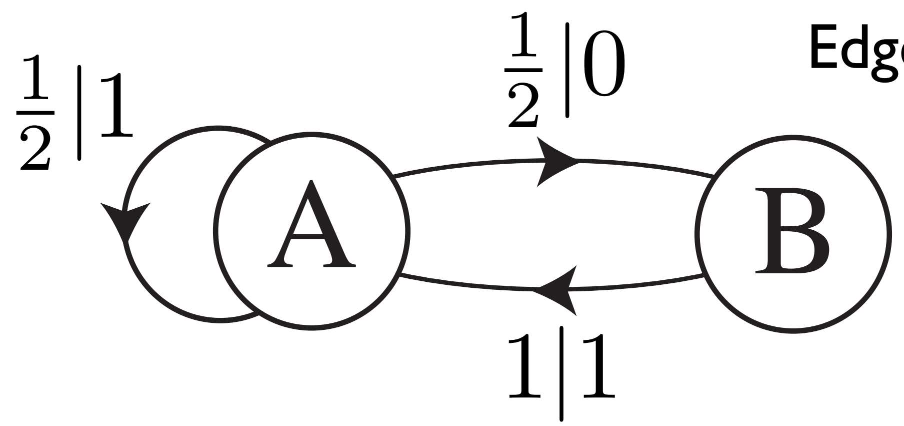
Classification of Discrete Stochastic Processes via Their Models:



From Determinism to Stochasticity ...

Two uses of HMMs:

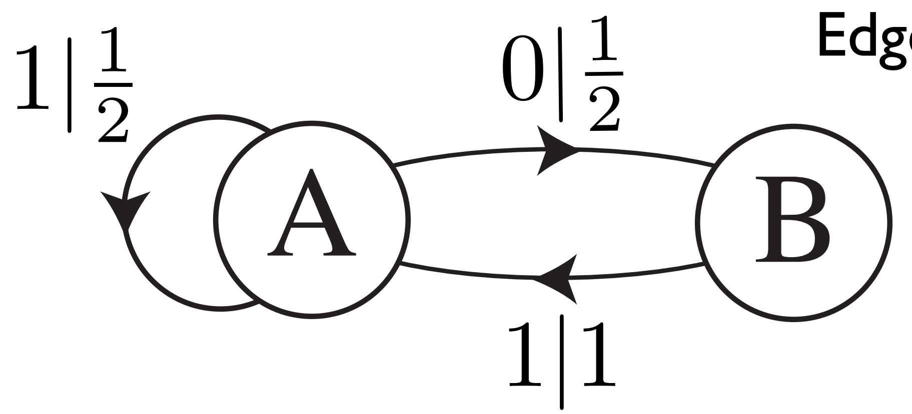
Generator:



Edge label: Transition probability | Symbol

Produces sequences, word distributions, ...

Recognizer:



Edge label: Symbol | Transition probability

Scan sequence, compare word distribution to a given distribution.
A sequence is probabilistically recognized when model assigns correct probability.

From Determinism to Stochasticity ...

Stochastic Processes ...

To calculate state distribution evolution

$$\vec{p}_V = (p_1, p_2, \ldots, p_k)$$

$$p_V(t+1) = p_V(t)\, T$$

z-Transform:

$$q_V(z) = \mathcal{Z}(p_V(t))$$

$$\mathcal{Z}(p_V(t)) = \sum_{t=0}^{\infty} p_v(t) z^{-t}$$

Inverse z-Transform:

$$p_V(t) = \mathcal{Z}^{-1}(q_V(z))$$

$$\mathcal{Z}^{-1}(q_V(z)) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz q_V(z) z^{t-1}$$

From Determinism to Stochasticity ...

Stochastic Processes ...

z-Transform ... $p_V(t+1) = p_V(t)\ T$

$$\sum_{t=0}^{\infty} p_V(t+1) z^{-t} = \sum_{t=0}^{\infty} p_V(t) T z^{-t}$$

$$\sum_{t=1}^{\infty} p_V(t) z^{-(t-1)} = q_V(z) T$$

$$z \left(\sum_{t=0}^{\infty} p_V(t) z^{-t} - p_V(0) \right) = q_V(z) T$$

$$z \left(q_V(z) - p_V(0) \right) = q_V(z) T$$

$$q_V(z) = \frac{p_V(0)}{(I - z^{-1} T)}$$

From Determinism to Stochasticity ...

Stochastic Processes ...

z-Transform Response Matrix:

$$R(t) = \mathcal{Z}^{-1}(\mathcal{T}(z)), \; \mathcal{T}(z) = (I - z^{-1}T)^{-1}$$

$$p_V(t) = \mathcal{Z}^{-1}(q_V(z)) = p_V(0)R(t)$$

$$p_V(t) = p_V(0)\; T^t$$

$$R(t) = T^t$$

$$(R(t))_{vv'} = \Pr(v', t | v, 0)$$

From Determinism to Stochasticity ...

Stochastic Processes ...

z-Transform Response Matrix ...

$$R(t) = A + B(t)$$

$$p_V(t) = p_V(0)A + p_V(0)B(t)$$

Asymptotic response matrix (time independent):

Recurrent or strongly connected states: $A_i = p_V(\infty), \ \forall i$

Multiply recurrent: $A_i \neq A_j$ $\quad p_V(0) = (0, \ldots, p_v = 1, \ldots, 0)$

$$A_v = p_V(\infty)$$

Transient response matrix (time dependent):

$$B(t) \rightarrow 0, \ t \rightarrow \infty$$

$$\sum_j B_{ij}(t) = 0$$

From Determinism to Stochasticity ...

Stochastic Processes ...

z-Transform Properties:

Fourier transform: $F(\omega) = \mathcal{F}(p_V(t)) = q_V(z = e^{2\pi i \omega})$

Linearity: $\mathcal{Z}(a p_V(t) + b p'_V(t)) = a q_V(z) + b q'_V(z)$

Time shift: $\mathcal{Z}(p_V(t - \tau)) = z^{-\tau} q_V(z)$

Scaling: $a^{-t} p_V(t) = q_V(az), a > 0$

Time reversal: $p_V(-t) = q_V(1/z)$

Convolution: $p_V(t) \star g(t) = q_V(z) \cdot G(z)$

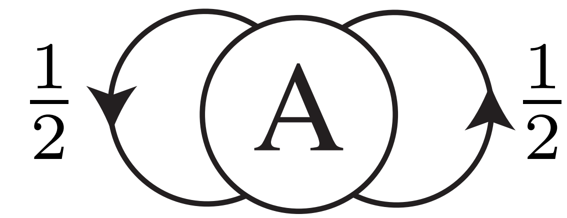
From Determinism to Stochasticity ...

Stochastic Processes ...

z-Transform Examples: $V = \{A\}$

$$T = (1)$$

$$p_V = (1)$$



$$q_V(z) = \mathcal{Z}(1) = \sum_{t=0}^{\infty} z^{-t} = \frac{1}{1 - z^{-1}}$$

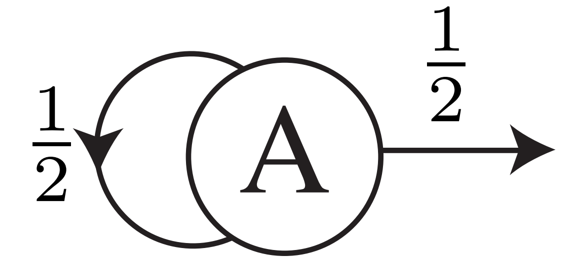
$$p_V(t) = \mathcal{Z}^{-1}\left(\frac{1}{1 - z^{-1}}\right)$$

$$= (2\pi i)^{-1} \int_{-\infty}^{\infty} dz \frac{z^{t-1}}{1 - z^{-1}}$$

$$= (2\pi i)^{-1}(2\pi i \cdot 1) = 1$$

Residue Theorem

From Determinism to Stochasticity ...
Stochastic Processes ...
z-Transform Examples: $V = \{A, \ldots\}$



$$p_A(t) = 2^{-t}$$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \ldots \\ 0 & \ddots & & & \\ 0 & & \ddots & & \\ \vdots & & \ldots & & \end{pmatrix}$$

$$q_V(z) = \mathcal{Z}(2^{-t}) = \sum_{t=0}^{\infty} (2z)^{-t} = \frac{2}{2 - z^{-1}}$$

$$p_V(t) = \mathcal{Z}^{-1}\left(\frac{2}{2 - z^{-1}}\right)$$

$$= (2\pi i)^{-1} \int_{-\infty}^{\infty} dz \frac{2z^{t-1}}{2 - z^{-1}}$$

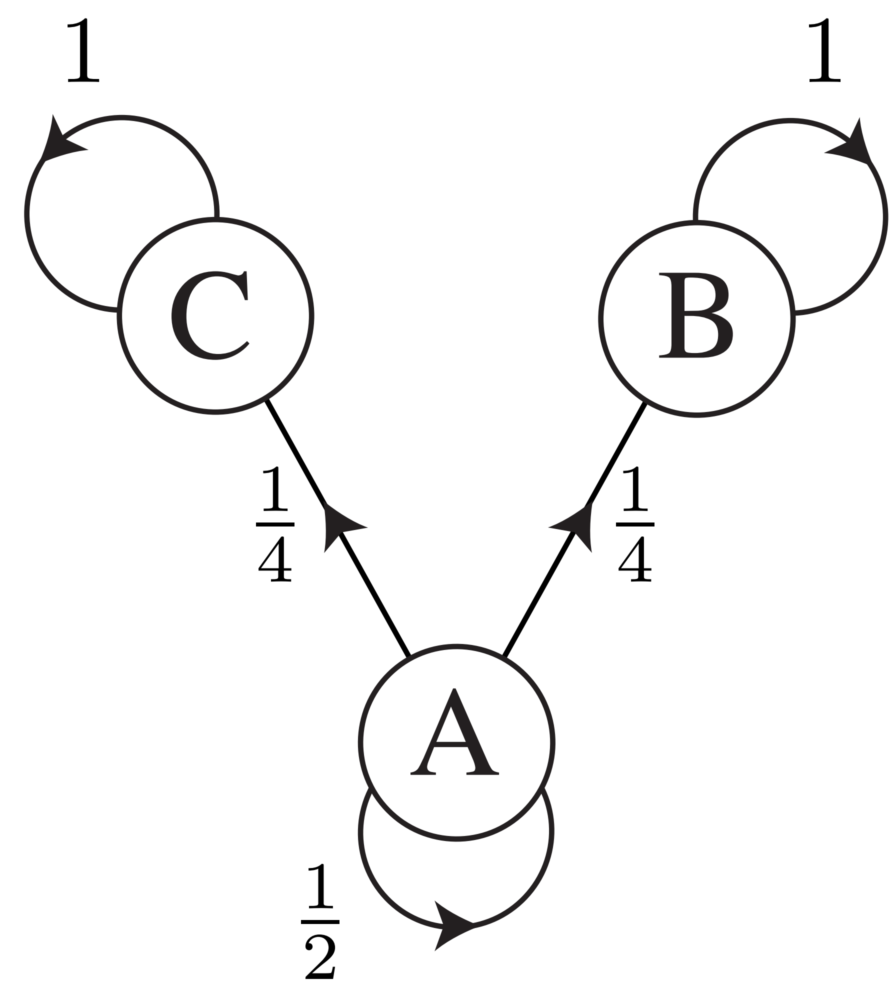
$$= (2\pi i)^{-1} \frac{1}{2} 2^{-(t-1)} \int_{-\infty}^{\infty} dz \frac{z^{t-1}}{1 - z^{-1}}$$

$$= (2\pi i)^{-1} (2\pi i \cdot 1) 2^{-t} = 2^{-t}$$

From Determinism to Stochasticity ...
Stochastic Processes ...
z-Transform Example:

$$V = \{A, B, C\}$$

$$T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



From Determinism to Stochasticity ...
Stochastic Processes ...
z-Transform Example ...

$$I - z^{-1}T = \begin{pmatrix} 1 - \frac{1}{2}z^{-1} & -\frac{1}{4}z^{-1} & -\frac{1}{4}z^{-1} \\ 0 & 1 - z^{-1} & 0 \\ 0 & 0 & 1 - z^{-1} \end{pmatrix}$$

$$\mathcal{T}(z) = (I - z^{-1}T)^{-1} = \begin{pmatrix} \frac{2}{2-z^{-1}} & \frac{z^{-1}}{2(1-z^{-1})(2-z^{-1})} & \frac{z^{-1}}{2(1-z^{-1})(2-z^{-1})} \\ 0 & \frac{1}{1-z^{-1}} & 0 \\ 0 & 0 & \frac{1}{1-z^{-1}} \end{pmatrix}$$

$$\mathcal{T}(z) = \frac{1}{1-z^{-1}} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{2}{2-z^{-1}} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(t) = \mathcal{Z}^{-1}(\mathcal{T}(z)) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2^{-t} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From Determinism to Stochasticity ...
Stochastic Processes ...
z-Transform Example ...

$$R(t) = \mathcal{Z}^{-1}(\mathcal{T}(z)) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2^{-t} \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$p_V(0) = (1, 0, 0)$$

$$p_V(t) = \left(2^{-t}, \tfrac{1}{2}(1 - 2^{-t}), \tfrac{1}{2}(1 - 2^{-t})\right)$$

$$p_V(\infty) = \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$p_V(0) = (0, 1, 0)$$

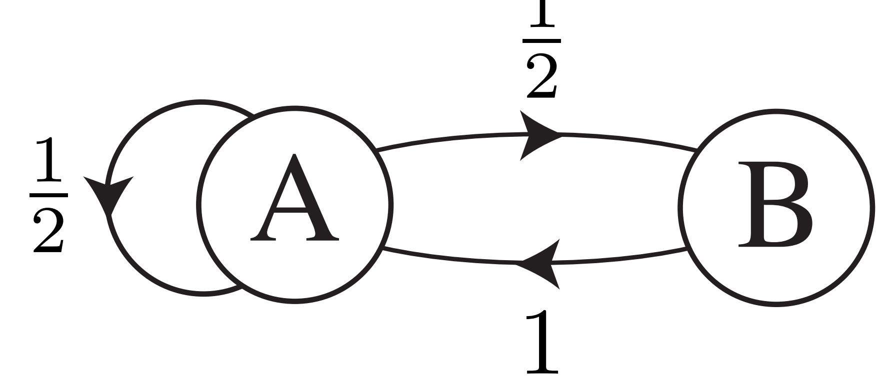
$$p_V(\infty) = (0, 1, 0)$$

From Determinism to Stochasticity ...

Stochastic Processes ...

z-Transform of Golden Mean Process:

$$V = \{A, B\} \quad T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$



$$I - z^{-1}T = \begin{pmatrix} 1 - \frac{1}{2}z^{-1} & -\frac{1}{2}z^{-1} \\ -z^{-1} & 1 \end{pmatrix}$$

$$\mathcal{T}(z) = (I - z^{-1}T)^{-1} = \begin{pmatrix} \frac{2z^2}{(2z+1)(z-1)} & \frac{z}{(2z+1)(z-1)} \\ \frac{2z}{(2z+1)(z-1)} & \frac{2z^2-z}{(2z+1)(z-1)} \end{pmatrix}$$

z-Transform of Golden Mean Process ...

$$R(t) = \begin{pmatrix} \mathcal{Z}^{-1}(\mathcal{T}_{00}) & \mathcal{Z}^{-1}(\mathcal{T}_{01}) \\ \mathcal{Z}^{-1}(\mathcal{T}_{10}) & \mathcal{Z}^{-1}(\mathcal{T}_{11}) \end{pmatrix}$$

$$R(t) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix} + \frac{1}{3}(\frac{-1}{2})^t \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$$

$$p_V(0) = (1, 0) \Rightarrow p_V(t) = \left(\frac{2}{3} + \frac{1}{3}(-\frac{1}{2})^t, \frac{1}{3} - \frac{1}{3}(-\frac{1}{2})^t\right)$$

$$p_V(\infty) = \left(\frac{2}{3}, \frac{1}{3}\right)$$

From Determinism to Stochasticity ...

Reading for next lecture:

Lecture Notes.