From Determinism to Stochasticity

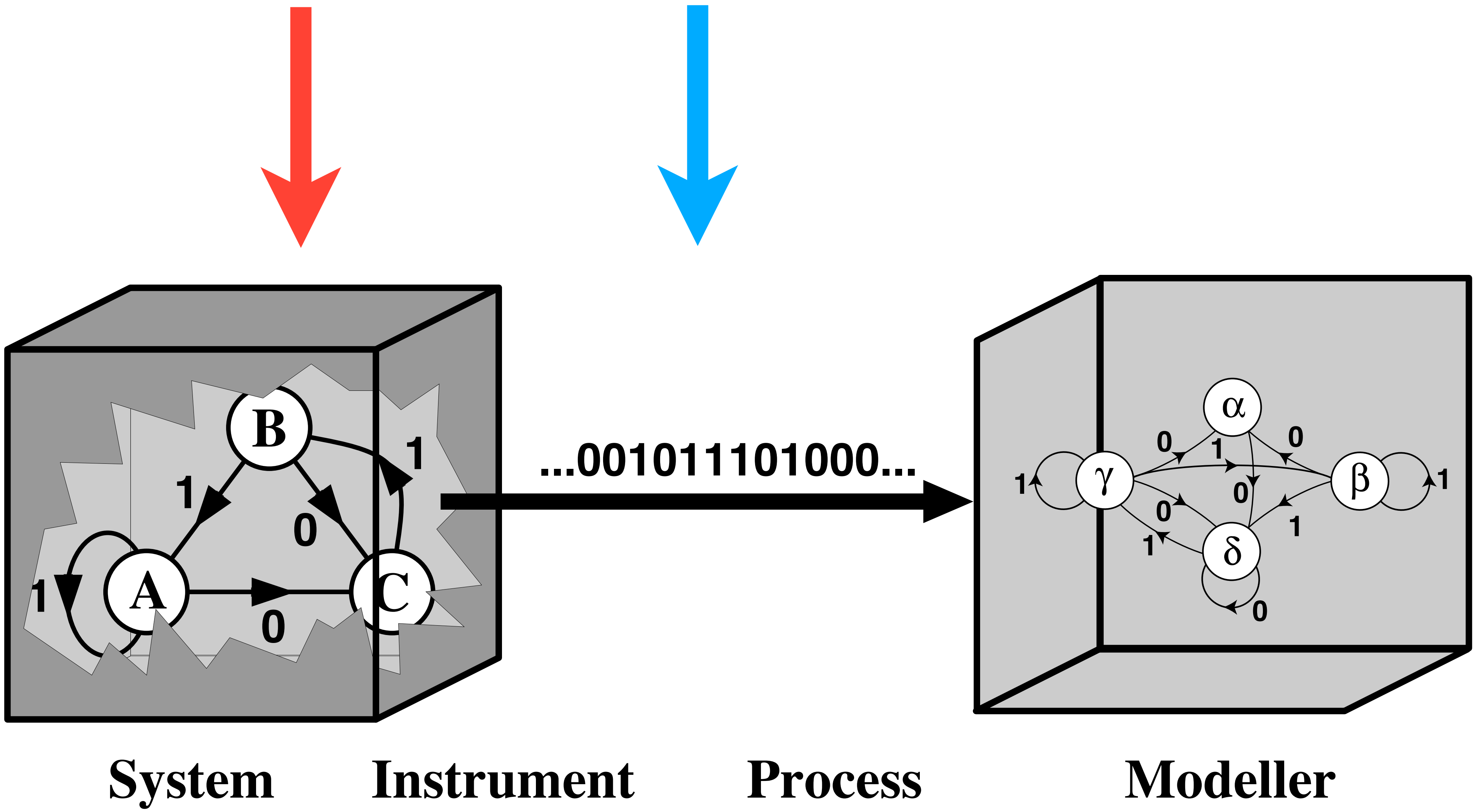
Reading for this lecture:

(These) *Lecture Notes*.

Outline of next few lectures:
- Probability theory
- Stochastic processes
- Measurement theory

You Are Here

You Will Soon Be Here



The Learning Channel

From Determinism to Stochasticity ...

Probability Theory of Dynamical Systems:

Probability Theory Review:

Discrete Random Variable (RV): X

Events (Alphabet): $\mathcal{X} = \{1, 2, \ldots, k\}$

Realization: $x \in \mathcal{X}$

Probability mass function ("distribution"): $\Pr(x) = \Pr\{X = x\}$

$$0 \leq \Pr(x) \leq 1, \;\; x \in \mathcal{X}$$

Normalized: $\displaystyle\sum_{x \in \mathcal{X}} \Pr(x) = 1$

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Probability Theory of Dynamical Systems:

Probability Theory Review ...

Discrete random variables:

1. Biased coin: $\mathcal{X} = \{H, T\}$

$$\Pr(H) = 1/3$$

$$\Pr(T) = 2/3$$

2. Sequence: No pairs of 0s

$$\mathcal{X} = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

$$\Pr(s^3) = \begin{cases} 0 & 000, 001, 100 \\ \frac{1}{3} & 101 \\ \frac{1}{6} & \text{otherwise} \end{cases}$$

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Probability Theory of Dynamical Systems:

Probability Theory Review ...

Continuous Random Variable: X

Takes values over continuous event space: $\mathcal{X}$

Cumulative distribution function: $P(x) = \Pr(X \leq x)$

$$0 \leq \Pr(x) \leq 1, \;\; x \in \mathcal{X}$$

If continuous, then random variable is.

Probability density function: $p(x) = P'(x) \quad 0 \leq p(x), \;\; x \in \mathcal{X}$

$$p(x)dx = \Pr(X < x + dx) - \Pr(X < x)$$

Normalization: $\Pr(X < \infty) = 1$ or $\int_{-\infty}^{\infty} dx \; p(x) = 1$

(e.g., if $x \in \mathbb{R}$)

Support of distribution: $\mathrm{supp} X = \{x : \; p(x) > 0\}$

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Probability Theory of Dynamical Systems:

Probability Theory Review ...

Continuous random variable X:

Uniform distribution on interval: $\mathcal{X} = \mathbb{R}$

Density: $p(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$

Distribution: $\Pr(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$

Support: $\text{supp } X = [0, 1]$

Probability Theory of Dynamical Systems:

Probability Theory Review ...

Continuous random variable X:

Gaussian: $\mathcal{X} = \mathbb{R}$

Density: $p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$

Distribution: $P(x) = \int_{-\infty}^{x} dy \; p(y) \equiv \mathrm{Erf}(x)$

Support: $\mathrm{supp} \; X = \mathbb{R}$

Probability Theory of Dynamical Systems:

Probability Theory Review ...

Discrete RVs: X over $\mathcal{X}$ & Y over $\mathcal{Y}$

Joint distribution: $\Pr(X, Y)$

Marginal distributions:

$$\Pr(X) = \sum_{y \in \mathcal{Y}} \Pr(X, y)$$

$$\Pr(Y) = \sum_{x \in \mathcal{X}} \Pr(x, Y)$$

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Probability Theory of Dynamical Systems:

Probability Theory Review ...

Factor joint distribution:

$$\Pr(X,Y) = \Pr(X|Y)\Pr(Y)$$

$$\Pr(X,Y) = \Pr(Y|X)\Pr(X)$$

Conditional distributions:

$$\Pr(Y|X) = \frac{\Pr(X,Y)}{\Pr(X)} \ , \ \Pr(X) \neq 0$$

$$\Pr(X|Y) = \frac{\Pr(X,Y)}{\Pr(Y)} \ , \ \Pr(Y) \neq 0$$

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Probability Theory of Dynamical Systems:

Probability Theory Review ...

Statistical independence: $X \perp Y$

$$\Pr(X, Y) = \Pr(X)\Pr(Y)$$

Conditional independence ("shielding"): $X \perp_Z Y$

$$\Pr(X, Y|Z) = \Pr(X|Z)\Pr(Y|Z)$$

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Dynamical Evolution of Distributions:

Dynamical system: $\{\mathcal{X}, \mathcal{T}\}$

State density: $p(x) \quad x \in \mathcal{X}$

Can evolve individual states and sets: $\mathcal{T} : x_0 \to x_1$

Initial density: $p_0(x)$ E.g., model of measuring a system

Evolve a density? $p_0(x) \to_{\mathcal{T}} p_1(x)$

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Dynamical Evolution of Distributions ...

Conservation of probability:

$$p_1(y)dy = p_0(x)dx$$

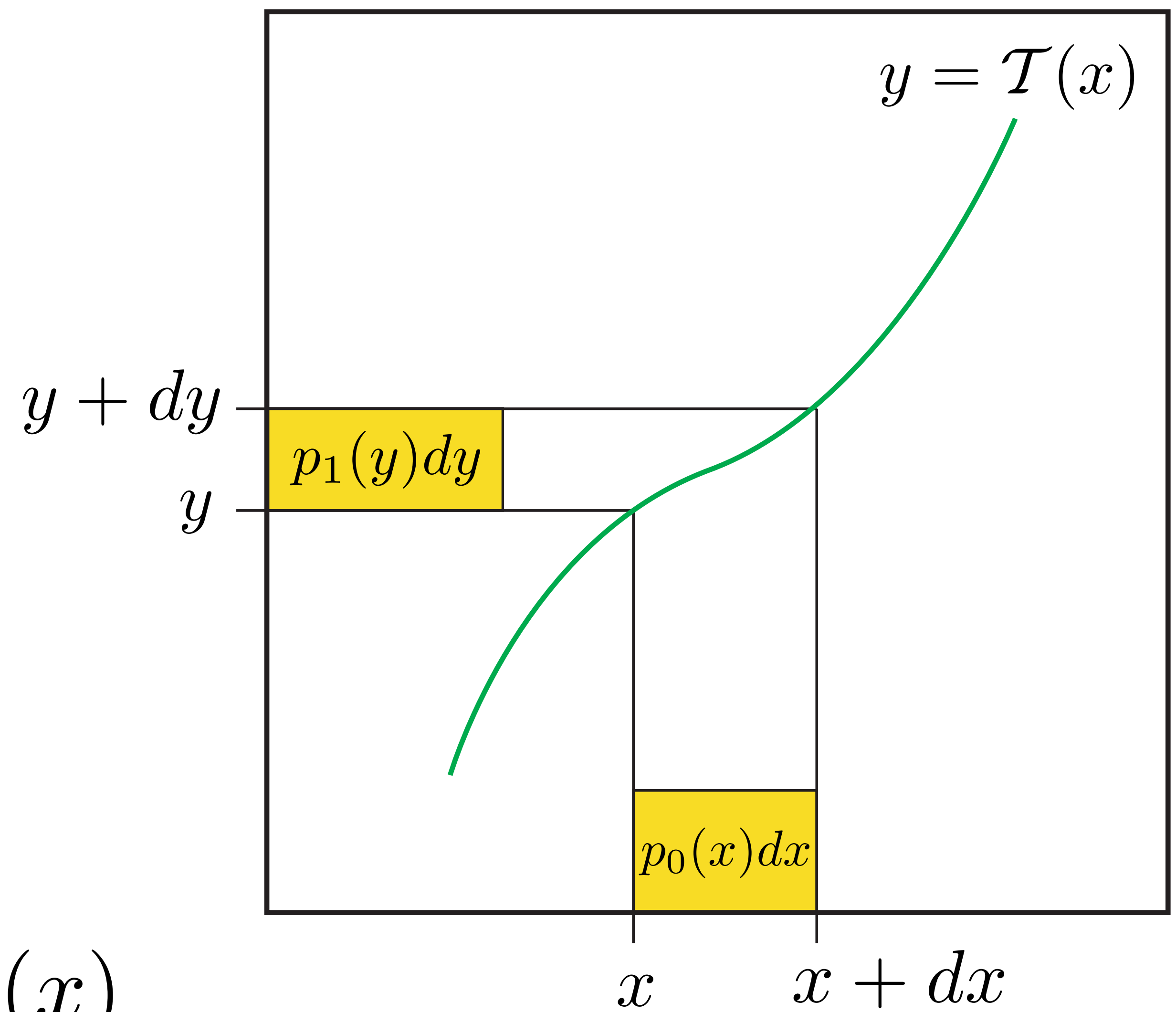
Perron-Frobenius Operator:

Locally: $y = \mathcal{T}(x)$

$$p_{n+1}(y) = \frac{p_n(x)}{|\mathcal{T}'(x)|}$$

Globally:

$$p_{n+1}(y) = \sum_{x \in \mathcal{T}^{-1}(y)} \frac{p_n(x)}{|\mathcal{T}'(x)|}$$



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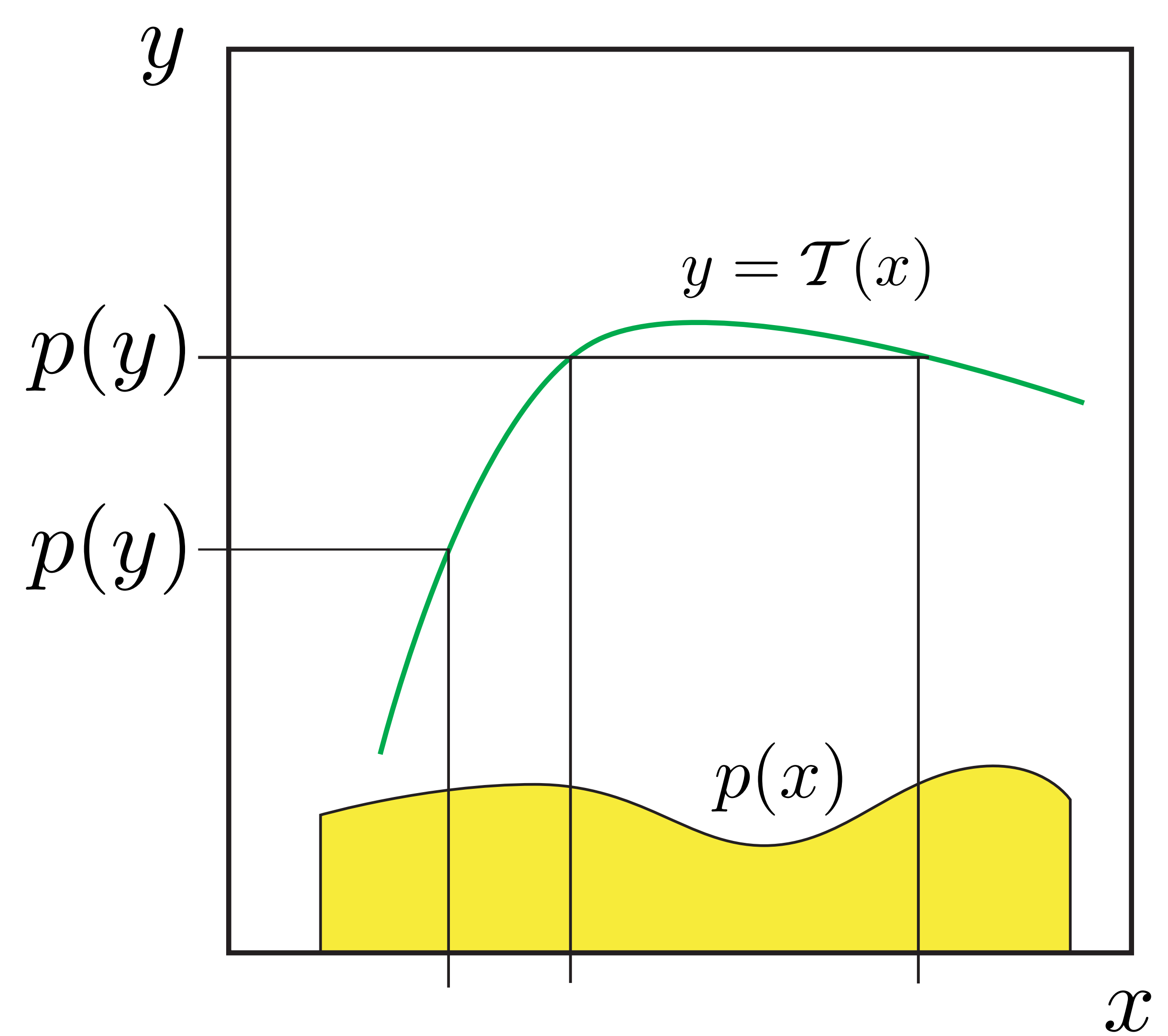
Dynamical Evolution of Distributions ...

Frobenius-Perron Equation:

$$p_{n+1}(y) = \int dx \; p_n(x) \delta(y - \mathcal{T}(x))$$



Dirac delta-function:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int dx \; \delta(x - c) f(x) = f(c)$$

$$\int dx \; \delta(x) = 1$$

Example: Delta function initial distribution

Map: $x_{n+1} = f(x_n)$

Initial condition: $x_0 \in \mathbf{R}$

Initial distribution: $p_0(x) = \delta(x - x_0)$

$$
\begin{aligned}
p_1(y) &= \int dx \; p_0(x) \; \delta(y - f(x)) \\
&= \int dx \; \delta(x - x_0) \; \delta(y - f(x)) \\
&= \delta(y - f(x_0)) \\
&= \delta(y - x_1) \\
&\vdots \\
p_n(y) &= \delta(y - x_n)
\end{aligned}
$$

... reduces to an orbit

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Probability Theory of Dynamical Systems ...

Dynamical Evolution of Distributions ...

Delta function IC: The easy case and expected result.

What happens when the IC has finite support?

$$p_0(x) = \begin{cases} 20, & |x - 1/3| \leq 0.025 \\ 0, & \text{otherwise} \end{cases}$$

Consider a set of increasingly more complicated systems and how they evolve distributions ...

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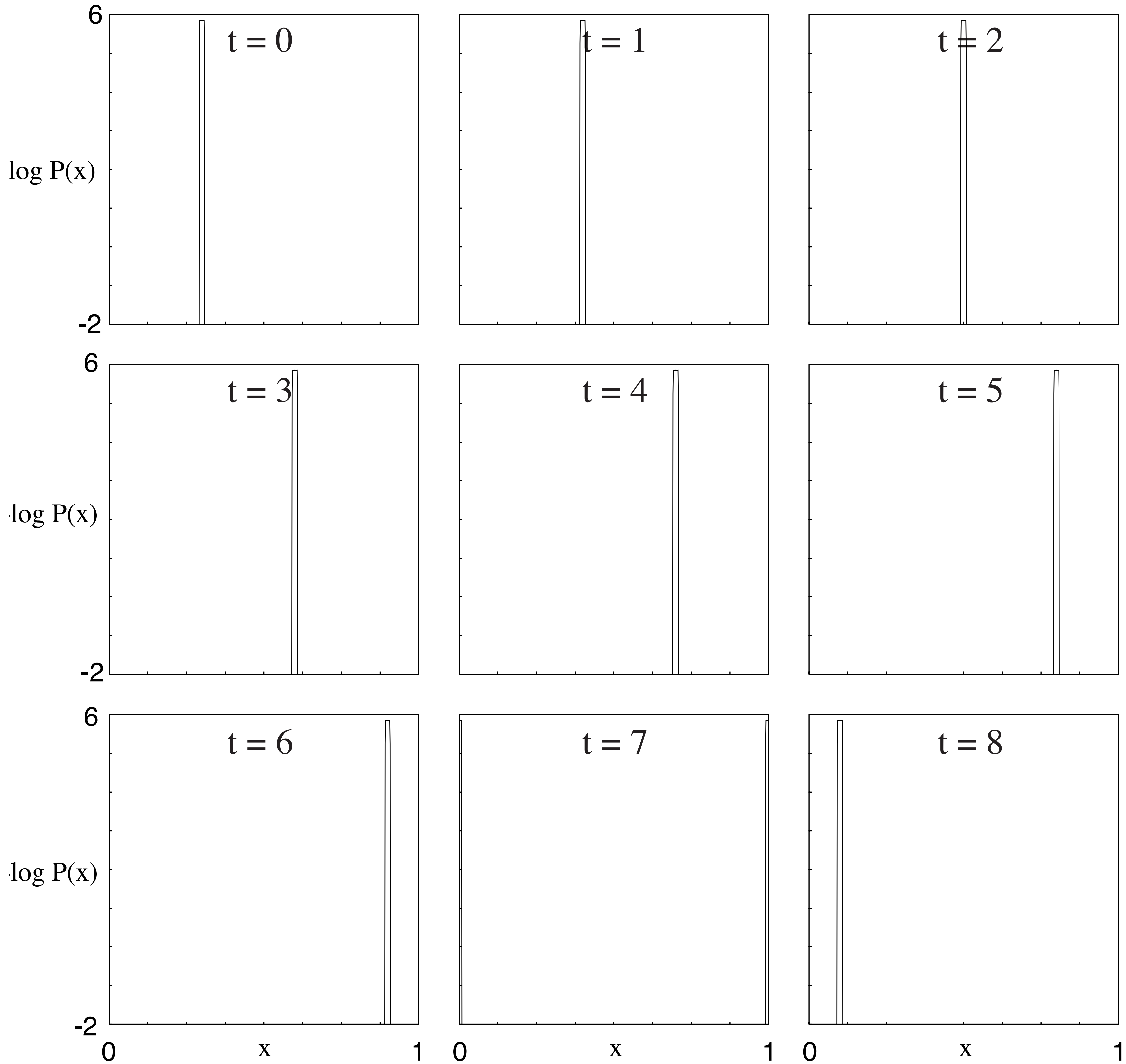
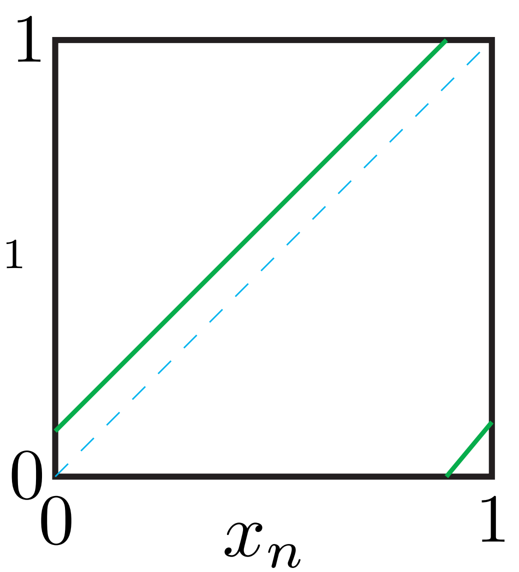
Dynamical Evolution of Distributions ...



Example:

Linear circle map

$$x_{n+1} = 0.1 + x_n \pmod 1$$



$$f'(x) = 1$$

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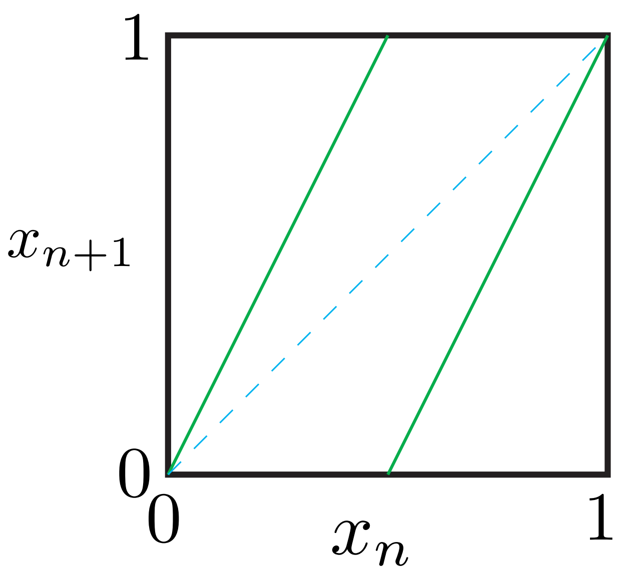
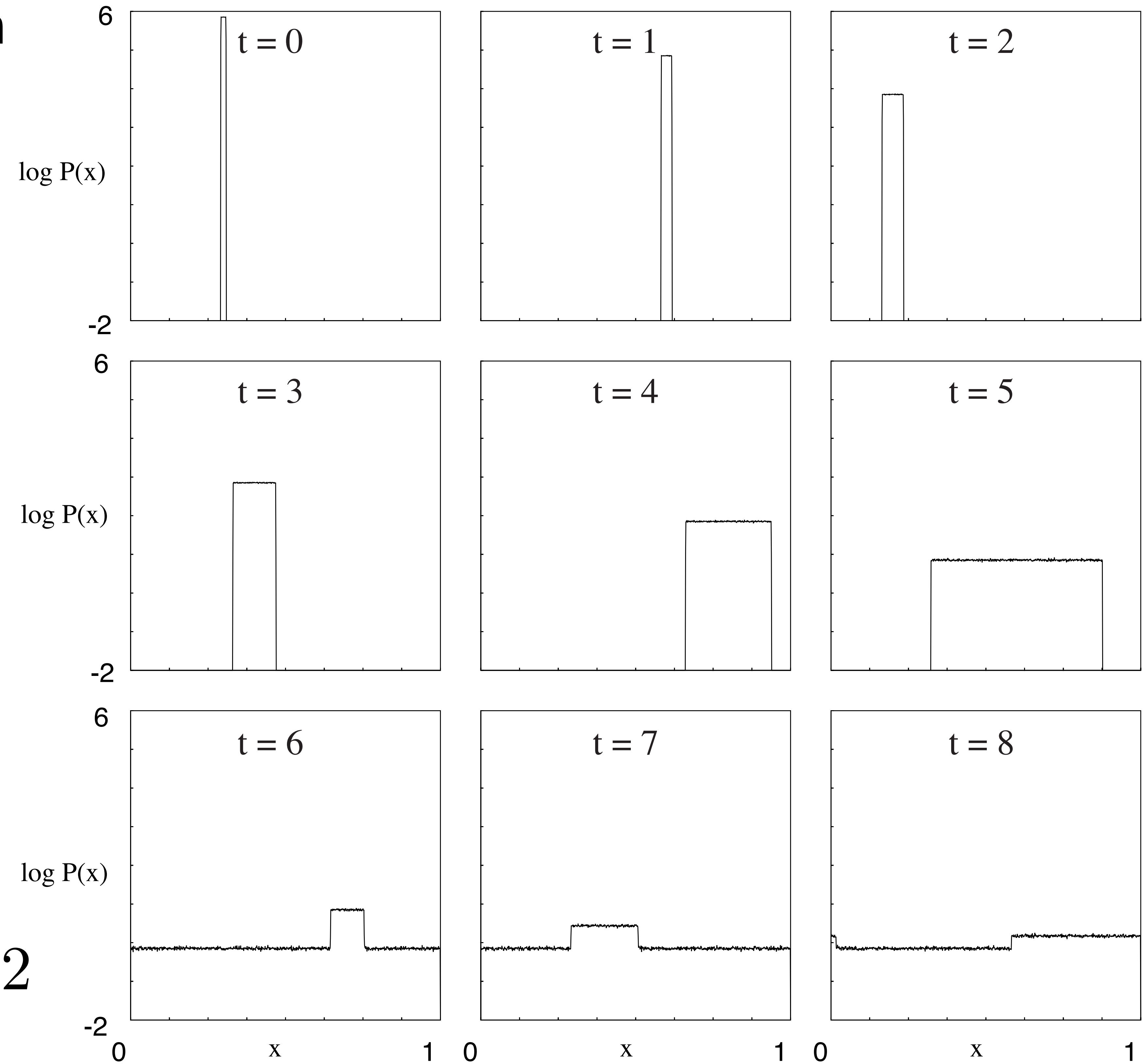
Dynamical Evolution of Distributions ...

Example:
Shift map





Spreading: $f'(x) = 2$

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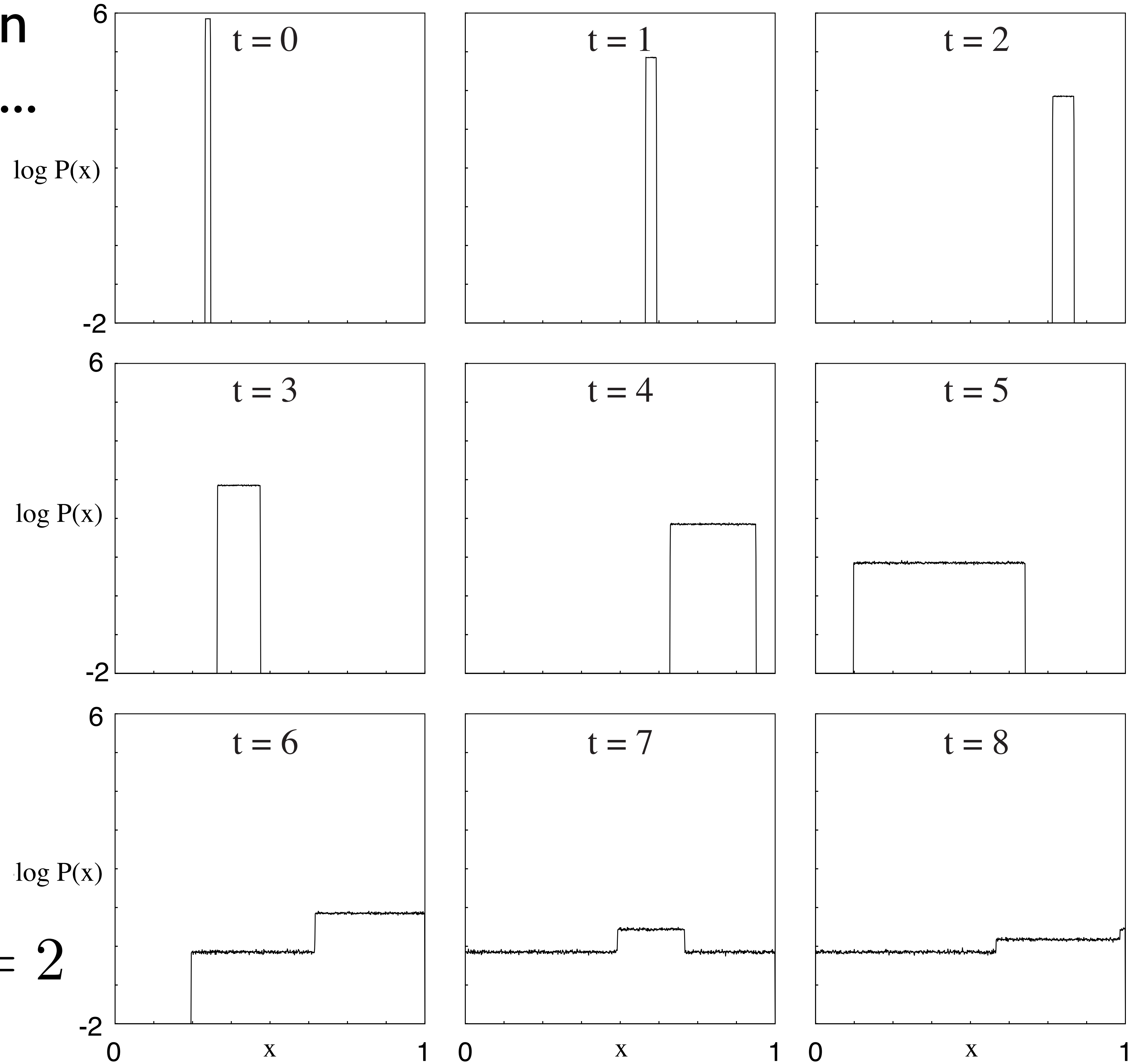
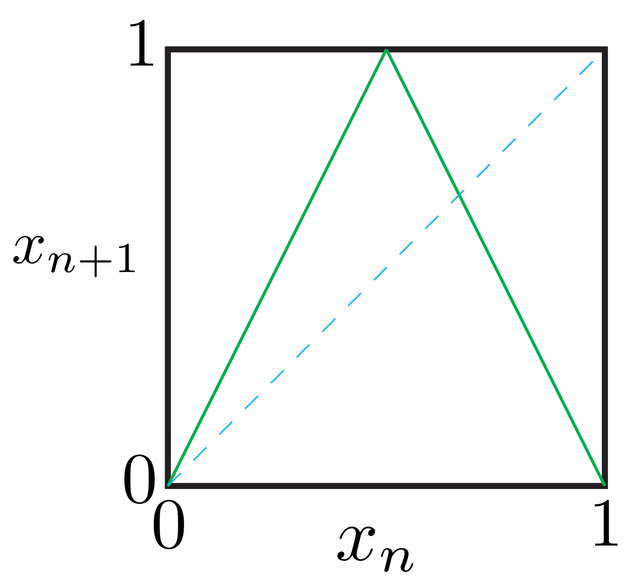


Example:

Tent map a = 2.0



Spreading: $|f'(x)| = 2$

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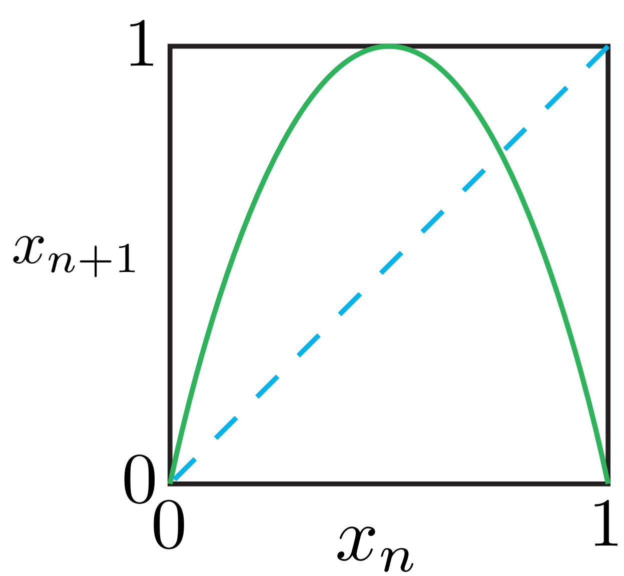
Example:

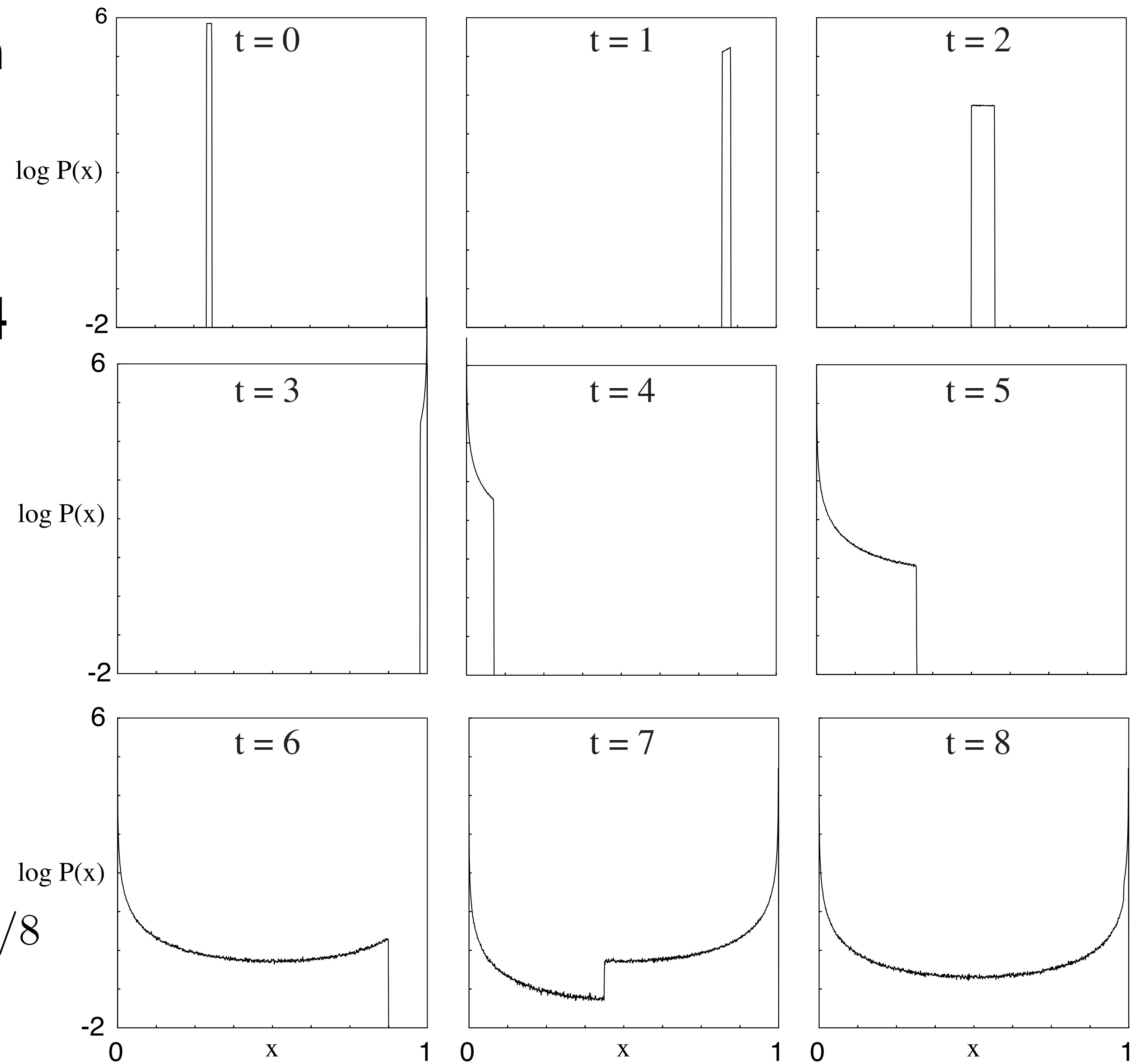
Logistic map r = 4



$$f'(x) = 4(1 - 2x)$$

Spreading: $x < 3/8$ or $x > 5/8$

Contraction: $3/8 < x < 5/8$



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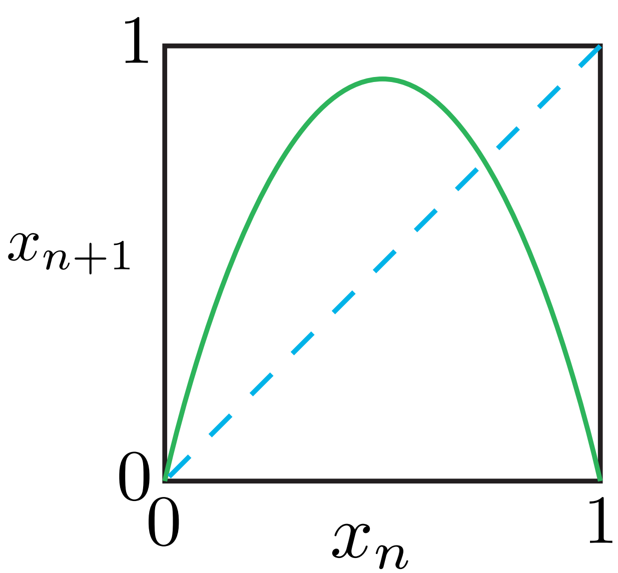
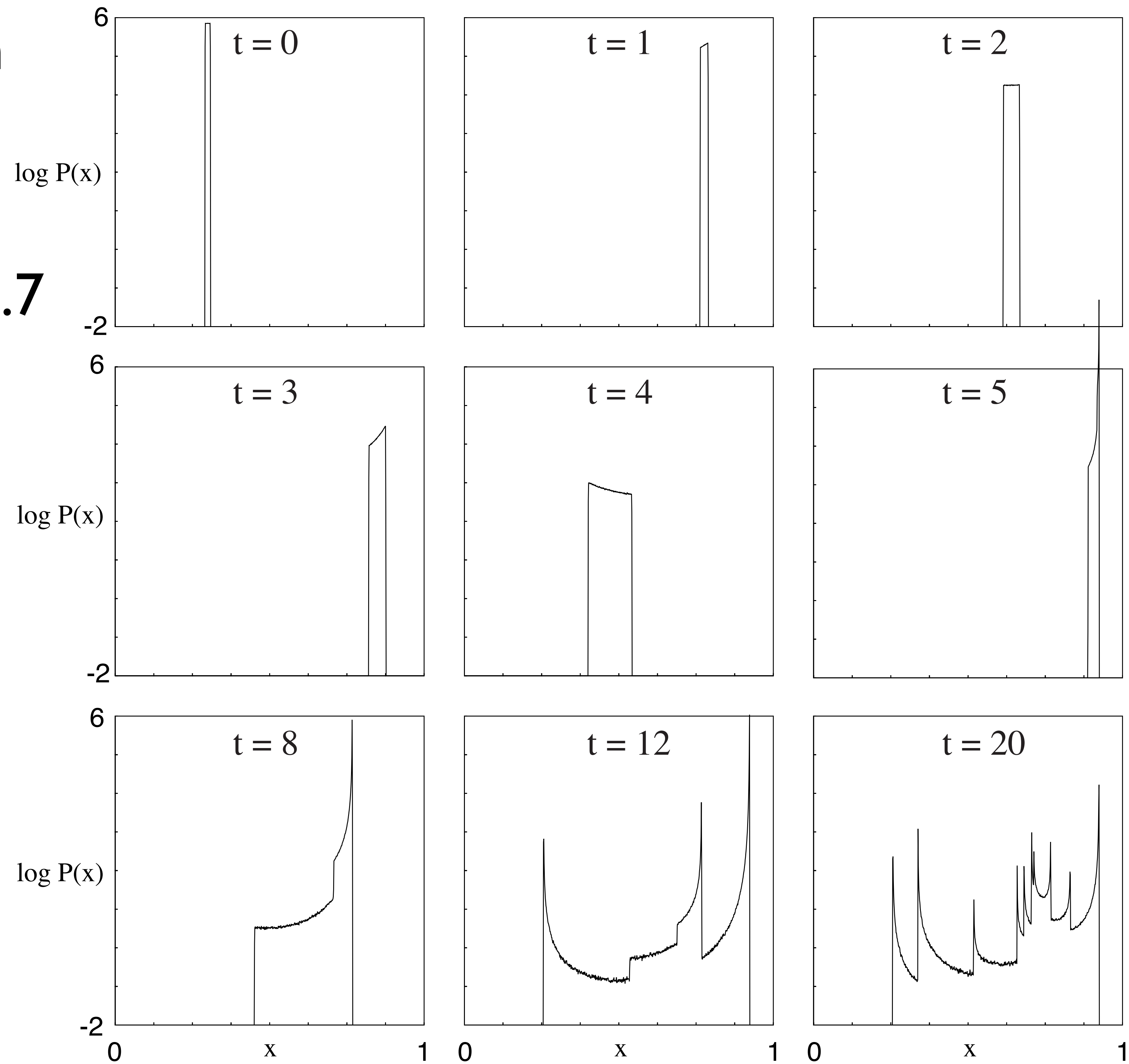
Example: Logistic map r = 3.7





Peaks in distribution are images of maximum

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Probability Theory of Dynamical Systems ...

Time-asymptotic distribution: What we observe

How to characterize?

Invariant measure:

A distribution that maps “onto” itself

Analog of invariant sets

Stable invariant measures:

Stable in what sense?

Robust to noise or parameters or ???

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Probability Theory of Dynamical Systems ...

Invariant measures for 1D Maps:

Probability distribution (density $p^*(x)$) that is invariant:

I. Distribution's support must be an invariant set:

$$\Lambda = f(\Lambda) \ , \quad \Lambda = \mathrm{supp}\ p^*(x) = \{x : p^*(x) > 0\}$$

II. Probabilities "invariant":

Distribution a fixed point of Frobenius-Perron Equation

$$p^*(y) = \int dx\ p^*(x)\ \delta(y - f(x))$$

Functional equation: Find $p^*(\cdot)$ that satisfies this.

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Example: Periodic-k orbit $\{x_1, x_2, \ldots, x_k\}$ has density

$$p(x) = \delta\left(\prod_{i=1}^{k}(x - x_i)\right)$$

Is it invariant?

$$p_1(y) = \int dx\ p(x)\delta(y - f(x))$$

$$= \int dx\ \delta\left(\prod_{i=1}^{k}(x - x_i)\right)\delta(y - f(x))$$

$$= \delta\left(\prod_{i=1}^{k}(y - f(x_i))\right)$$

$$= \delta\left(\prod_{i=1}^{k}(y - x_{(i+1) \bmod k})\right)$$

$$= \delta\left(\prod_{i=1}^{k}(y - x_i)\right)$$

Yes!

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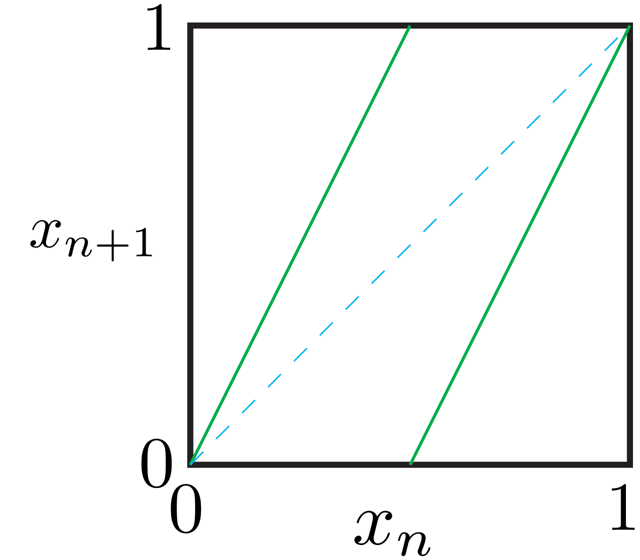
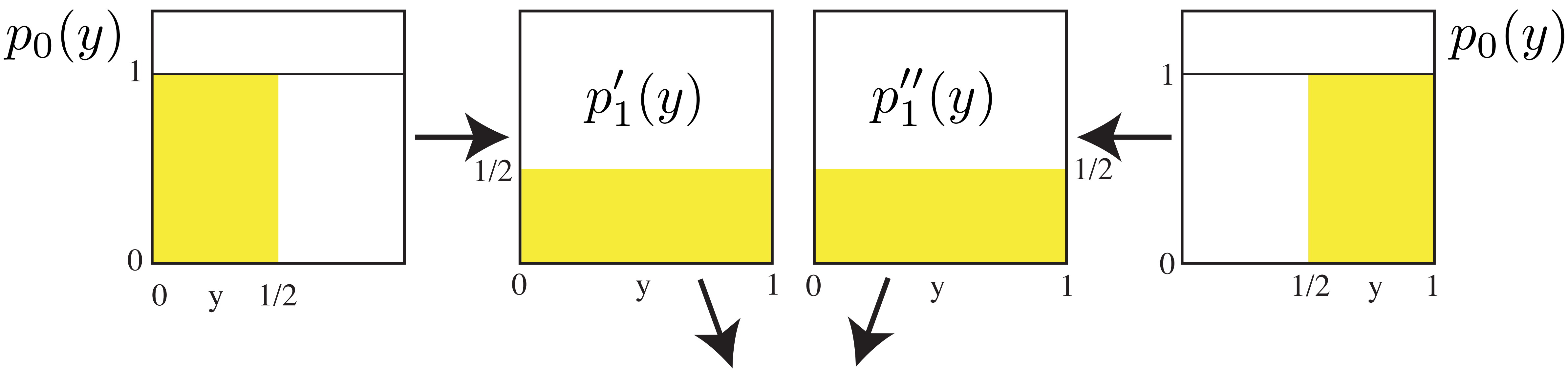
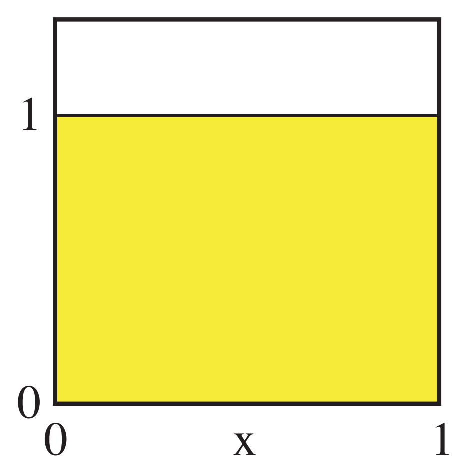
Example: Shift map invariant distribution

Uniform distribution: $p(x) = 1, \; x \in [0, 1]$







$$p_1(y) = p_1'(y) + p_1''(y)$$

$$p_1(x) = p_0(x) \qquad (y \Rightarrow x)$$

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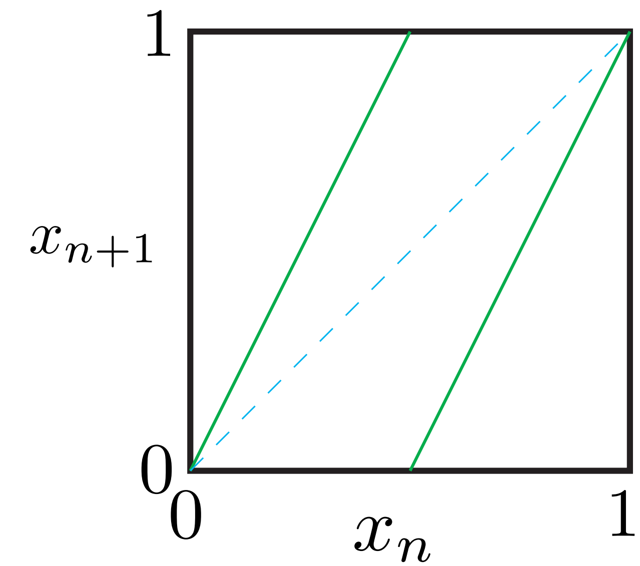
Example: Shift map invariant distribution

Uniform distribution: $p(x) = 1, \; x \in [0,1]$



Via Frobenius-Perron Equation: Two cases

A: $0 \leq x \leq 1/2$

$$\begin{aligned} p_1'(y) &= \int_0^{\frac{1}{2}} dx \; p_0(x) \delta(y - f(x)) \\ &= \int_0^{\frac{1}{2}} dx \; \delta(y - 2x) \\ &= \frac{1}{2} \end{aligned}$$

B: $1/2 < x \leq 1$

$$\begin{aligned} p_1''(y) &= \int_{\frac{1}{2}}^{1} dx \; p_0(x) \delta(y - f(x)) \\ &= \int_{\frac{1}{2}}^{1} dx \; \delta(y - 2x) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} p_1(y) &= p_1'(y) + p_1''(y) \\ &= p_0(x) \quad (y \Rightarrow x) \end{aligned}$$

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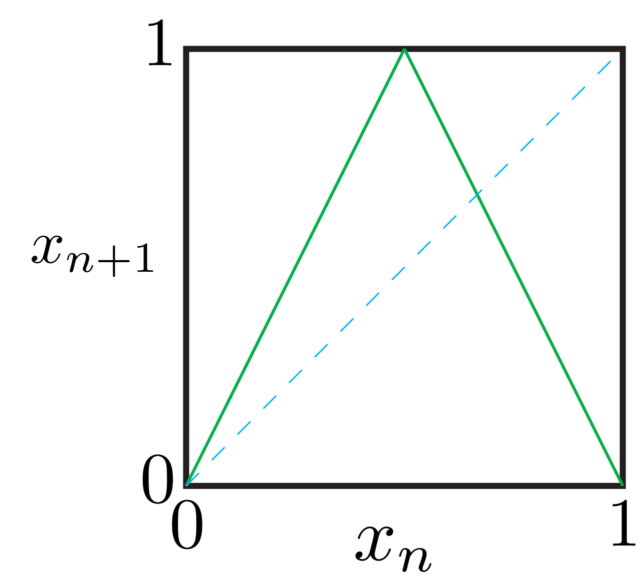
Probability Theory of Dynamical Systems ...

Example: Tent map $$x_{n+1} = \begin{cases} a x_n, & 0 \leq x_n \leq \frac{1}{2} \\ a(1 - x_n), & \frac{1}{2} < x_n \leq 1 \end{cases}$$

Fully two-onto-one: $a = 2$

Uniform distribution is invariant: $p(x) = 1, \; x \in [0, 1]$

Proof from FP Equation: Two cases



First case: exactly that of shift map

Second case: |slope| is all that's important

$1/2 < x \leq 1$

$$p_1''(y) = \int_{\frac{1}{2}}^{1} dx \; p_0(x) \delta\left(y - f(x)\right)$$

$$= \int_{\frac{1}{2}}^{1} dx \; \delta\left(y - (2 - 2x)\right) = \frac{1}{2}$$

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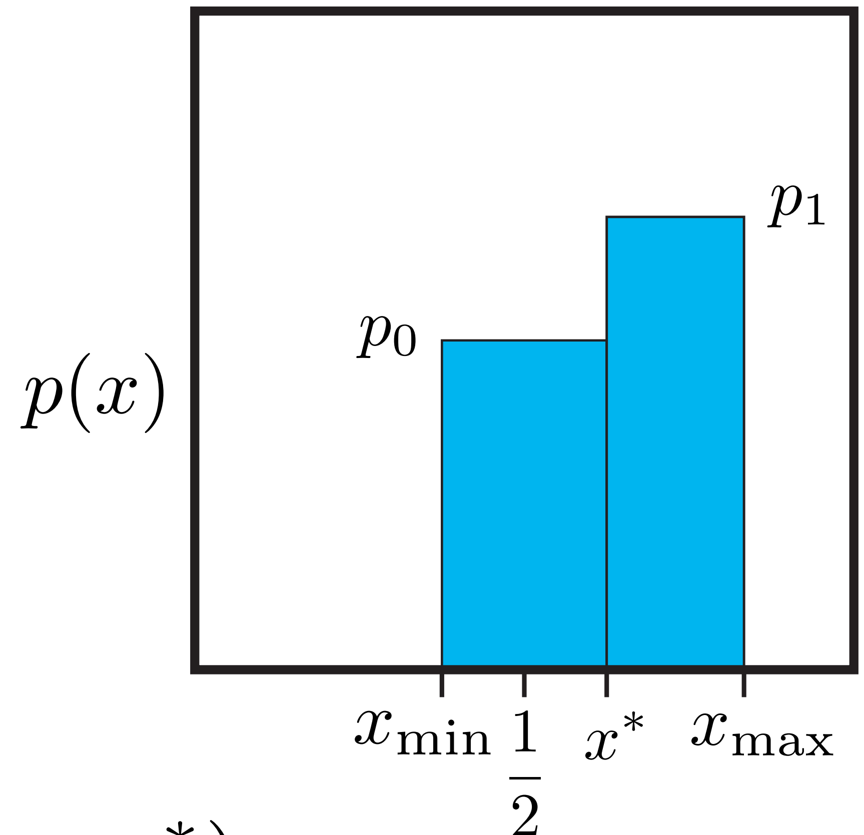
Probability Theory of Dynamical Systems ...

Example: Tent map where two bands merge to one: $a = \sqrt{2}$

Invariant distribution:

$$p(x) = \begin{cases} p_0, & x_{\min} \leq x \leq x^* \\ p_1, & x^* < x \leq x_{\max} \\ 0, & \text{otherwise} \end{cases}$$



$$x_{\max} = a/2$$
$$x_{\min} = a(1 - a/2)$$
$$x^* = a/(1 + a)$$

Equal areas: $p_0(x^* - x_{\min}) = p_1(x_{\max} - x^*)$

Normalization: $p_0(x^* - x_{\min}) + p_1(x_{\max} - x^*) = 1$

$$p_0 = \frac{1}{2(x^* - x_{\min})} \qquad p_1 = \frac{1}{2(x_{\max} - x^*)}$$

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Probability Theory of Dynamical Systems ...

Example: Logistic map $x_{n+1} = rx_n(1 - x_n)$

Fully two-onto-one: $r = 4$

Invariant distribution? $p(x) = \dfrac{1}{\pi\sqrt{x(1 - x)}}$

Exercise.

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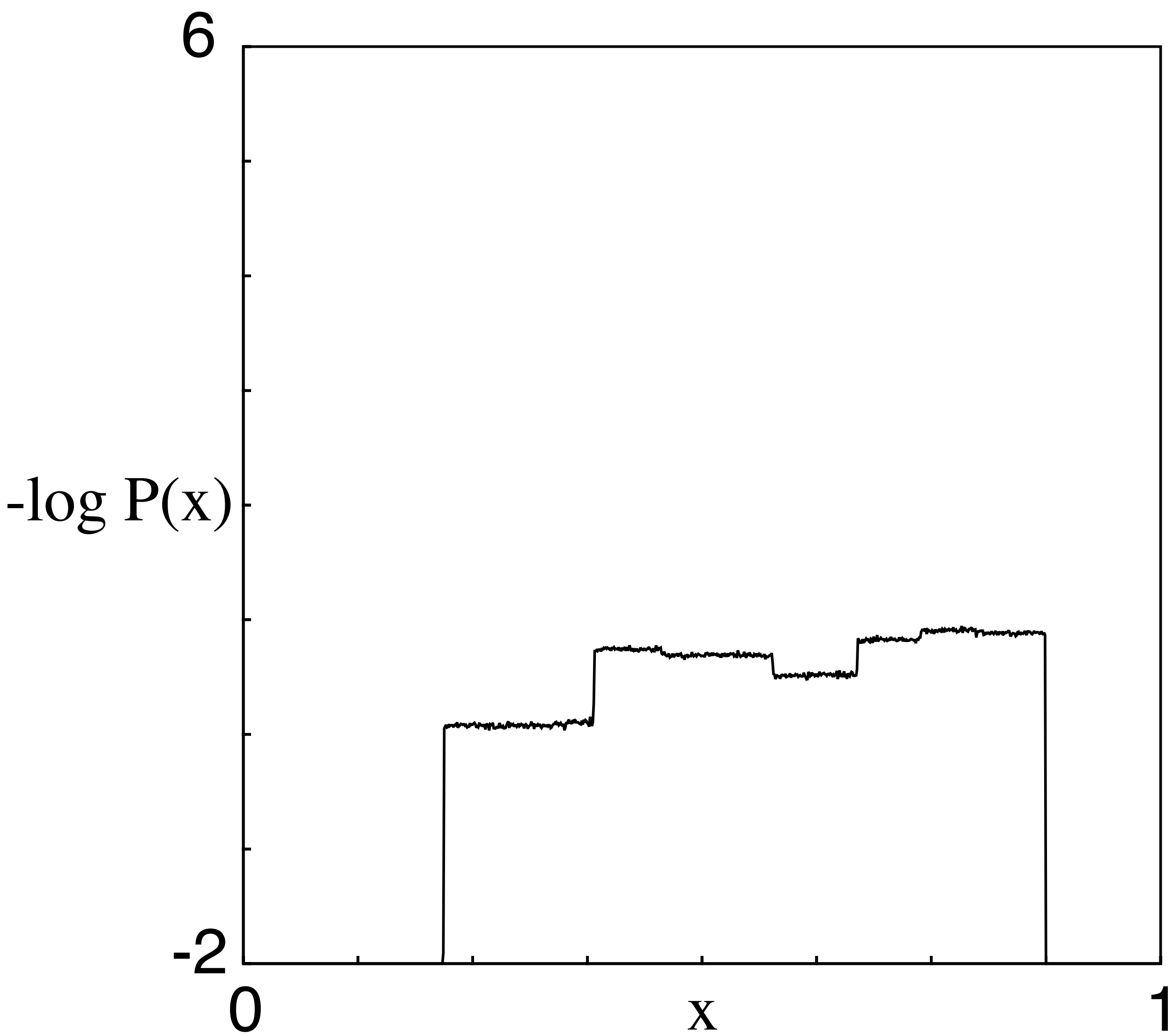
Numerical Example: Tent map

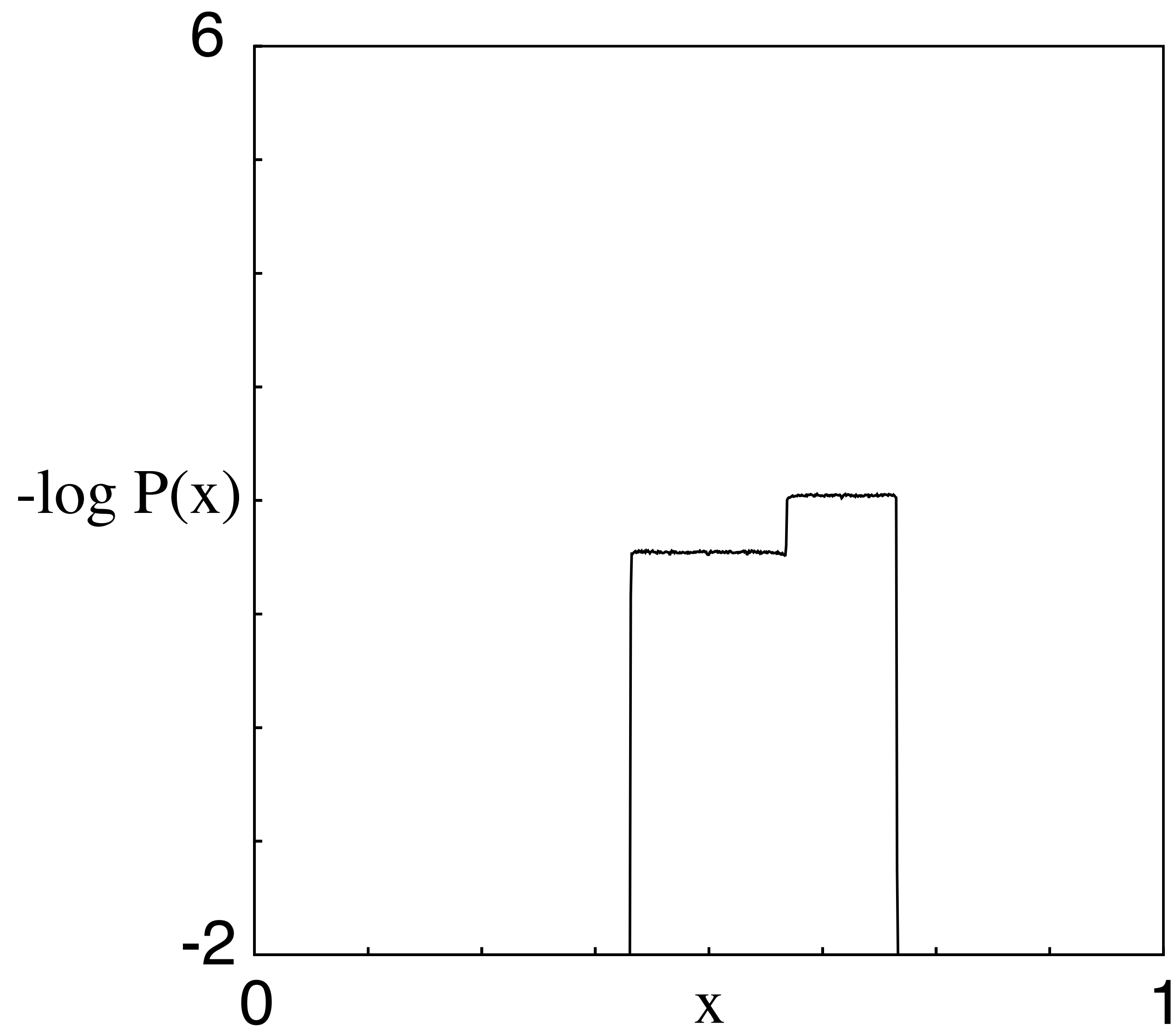
Typical chaotic parameter:

$$a = 1.75$$



Two bands merge to one:

$$a = \sqrt{2}$$



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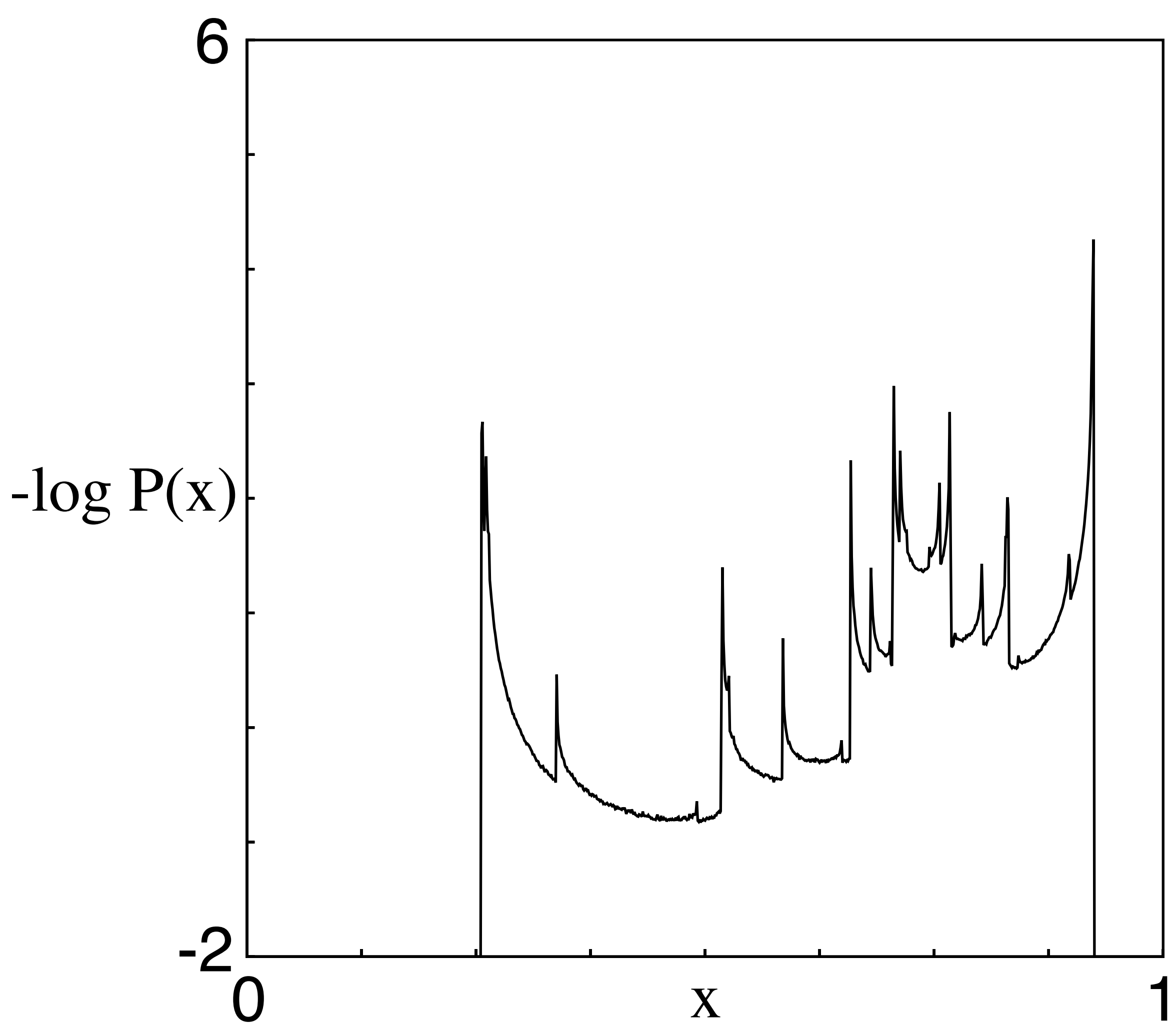
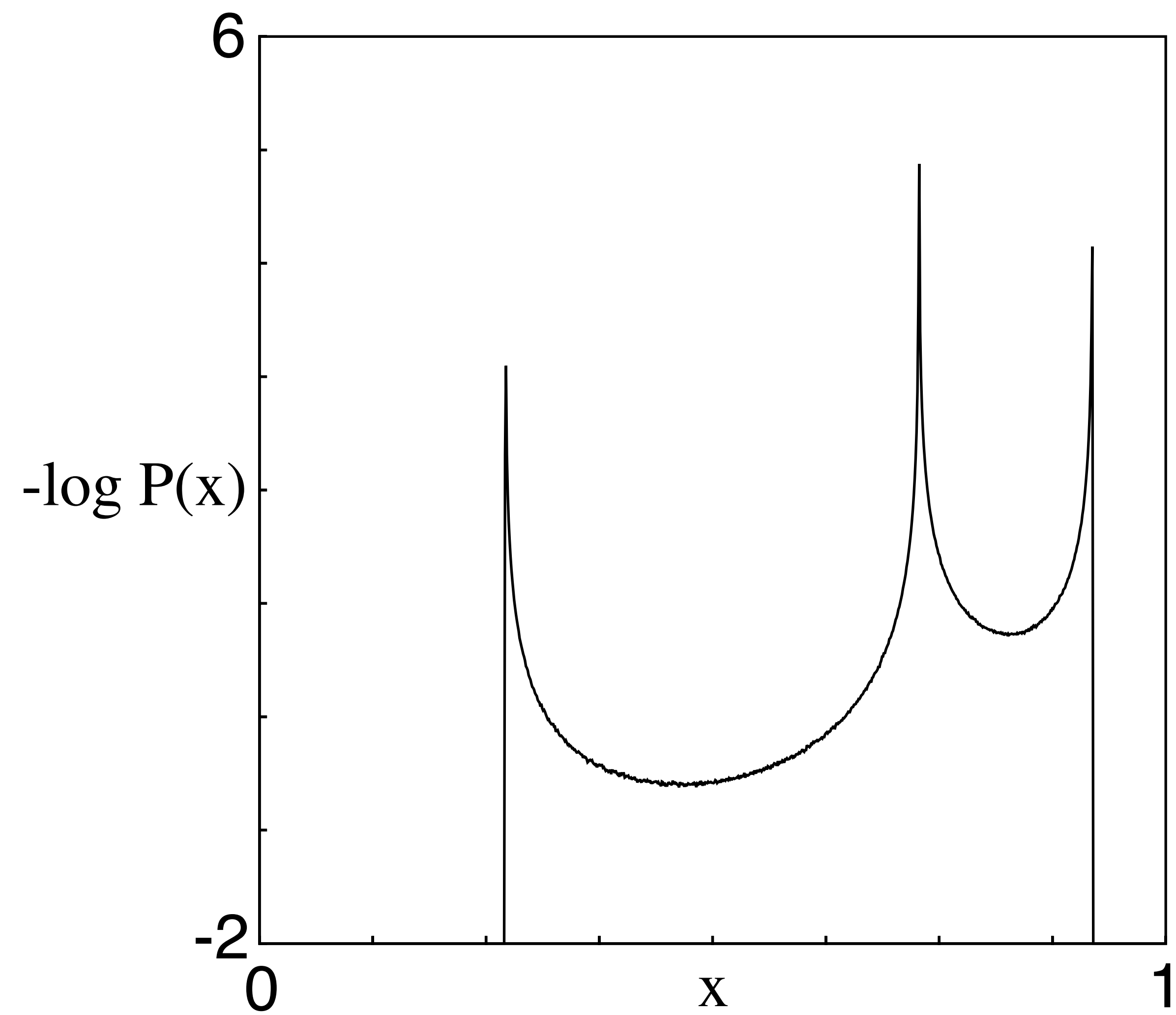
Probability Theory of Dynamical Systems ...

Numerical Example: Logistic map $x_{n+1} = rx_n(1 - x_n)$

Typical chaotic parameter:
$r = 3.7$



Two bands merge to one:
$r = 3.6785735104283219$



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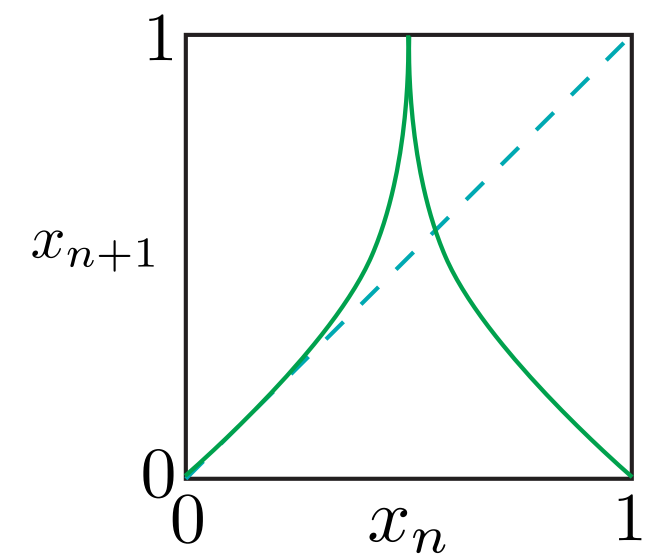
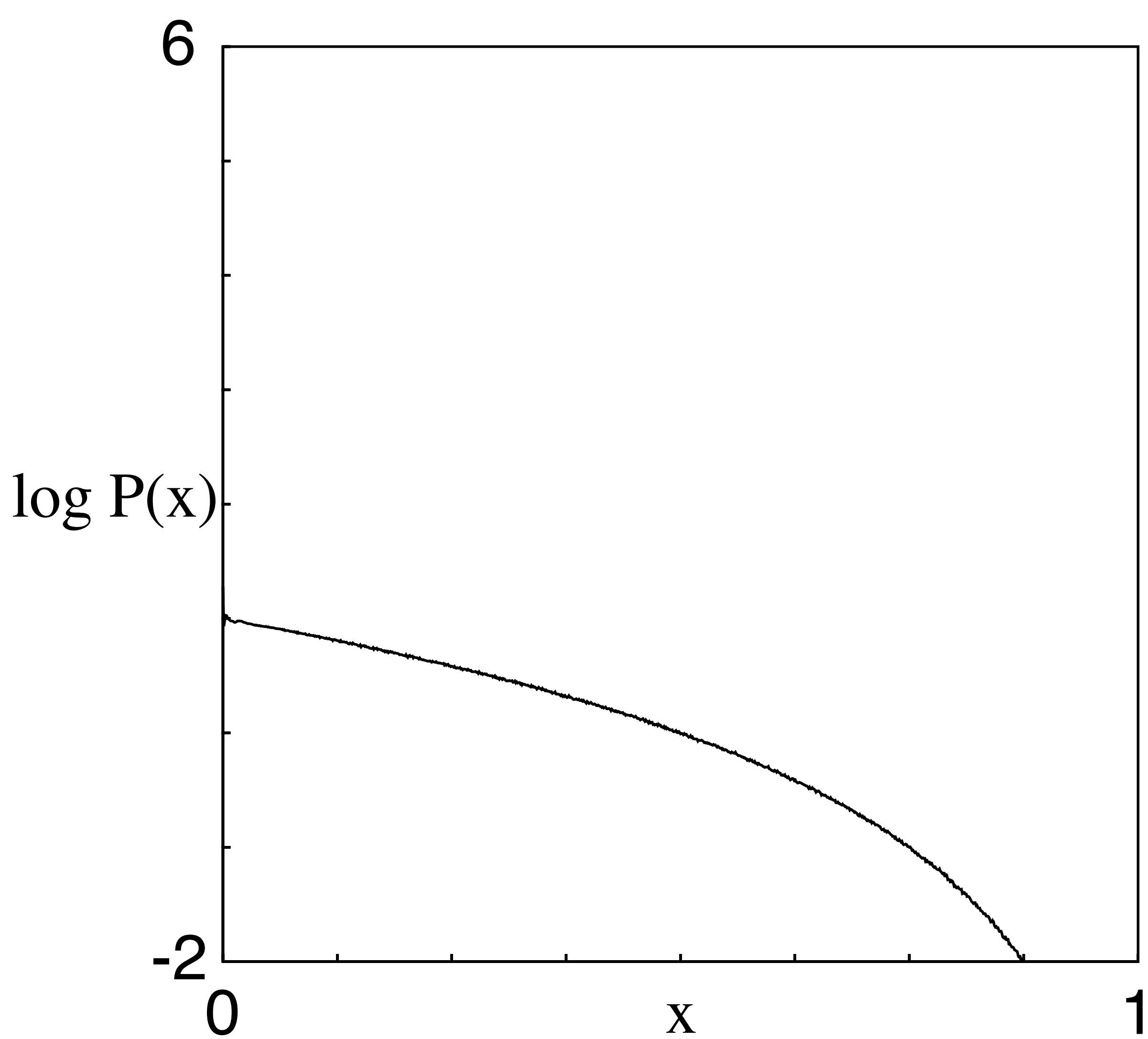
Numerical Example: Cusp map $x_{n+1} = a(1 - |1 - 2x_n|^b)$

$(a, b) = (1, 1/2)$





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Probability Theory of Dynamical Systems ...

Issue: Many invariant measures in chaos:

An infinite number of unstable periodic orbits: Each has one.
But none of these are what one sees,
one sees the aperiodic orbits.

How to exclude periodic orbit measures?

Add noise and take noise level to zero; which measures are left?

Robust invariant measures.

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Reading for next lecture:

Lecture Notes.