Falling cat problem

Yoshihisa Sakurai
Department of Mechanical and Aeronautical Engineering
University of California, Davis
yosakurai@ucdavis.edu

June 9, 2009

Abstract

The purpose of this paper is to examine the behavior of a falling cat when a torque is applied its joint. To do this, the cat was modeled by two cylinders connected by a ball-socket-joint. The implementation was carried out by changing coefficients of the torque applied the joint.

Introduction

It is well known that the cat released with upside-down orientation lands on its feet by cat righting reflex which is a cat's innate ability to land on its feet when it is released in any orientation[1]. Falling cat problem is to explain the underlying physics behind this phenomena. How a free-falling cat can turn itself as it falls without violating the law of conservation of angular momentum is a big topic of this problem[2] and many studies have been done in the point of view of mathematics and dynamics. In dynamic research, a mechanical model to represent a falling cat was established[3]. Then, an optimal joint angle trajectories of a falling cat released with upside-down orientation was calculated using this model[4]. As a mathematical approach, a falling cat theorem consisting of gauge theory of Yang-Mills field was established[5]. As done in previous studies, this problem is including a control theory for nonholonomic systems, an optimization of motion, Lagrangian reduction, differential geometry, and the gauge theory of Yang-Mills field. Falling cat problem is, therefore, very interesting topic to analyze.

In previous studies, the researchers used only joint angels to analyze. All animals generally, however, move by torque created by their muscles if there is no external force and torque. It is, therefore, important the influence of torque on cat's orientation in space.

The purpose of this project is to investigate the effects of torque at a ball-socket joint connecting two rigid bodies. This is achieved by deriving equations of motion of a free falling cat with giving constant torques at joint.

Dynamical system

System of falling cat

To simplify the falling cat model, two uniform cylinders had same size and weight(Figure 1). First, the inertia frame and two body fixed frame were defined:

$\hat{n}_1$: to the right
$\hat{n}_2$: vertically upwards
$\hat{n}_3 \equiv \hat{n}_1 \times \hat{n}_2$
$\hat{a}_1$: from the ball-socket-joint to the A*
$\hat{a}_2$: perpendicular to $\hat{a}_1$and upwards in $\hat{n}_1$and $\hat{n}_2$ plane in initial position
$\hat{a}_3 \equiv \hat{a}_1 \times \hat{a}_2$
$\hat{b}_1$: from the B* to the ball-socket-joint
$\hat{b}_2$: perpendicular to $\hat{b}_1$and upwards in $\hat{n}_1$and $\hat{n}_2$ plane in initial position
$\hat{b}_3 \equiv \hat{b}_1 \times \hat{b}_2$

The following nine generalized coordinates were chosen to describe the configuration of the system:

- $q_i(i = 1, 2, 3)$: The displacements in the $\hat{n}_1$, $\hat{n}_2$, and $\hat{n}_3$ directions of the center of mass of body A from the origin of the inertia reference frame.
- $q_i(i = 4, 5, 6)$: The angular displacements of successive rotations about the body fixed coordinates of A relative to Newtonian reference frame. Starting with the A reference frame aligned with the Newtonian reference frame. First rotate $\hat{q}_4$ about the $\hat{a}_1$ axis. Next rotate $\hat{q}_5$ about the new $\hat{a}_2$ axis. Finally, rotate $\hat{q}_6$ about the new $\hat{a}_3$ axis. These are the Euler's angles that the reference frame A relative to the Newtonian frame.
- $q_i(i = 7, 8, 9)$: The angular displacements of successive rotations about the body fixed coordinates of B relative to the reference frame A. Starting with the reference frame B aligned with the reference frame A. First rotate $\hat{q}_7$ about the $\hat{b}_1$ axis. Next rotate $\hat{q}_8$ about the new $\hat{b}_2$ axis. Finally, rotate $\hat{q}_9$ about the new $\hat{b}_3$ axis. These are the Euler's angles that the reference frame B relative to the reference frame A.

Additionally, the following nine generalized speeds were used:

$$u_i = v^{A^*} \cdot \hat{n}_j(i,\, j = 1, 2, 3)$$

$$u_i = \omega^{A^*N} \cdot \hat{n}_j(i = 4, 5, 6;\, j = 1, 2, 3)$$

$$u_i = \omega^{B^*N} \cdot \hat{n}_j(i = 7, 8, 9;\, j = 1, 2, 3)$$

In this project, a falling cat produced an interactive torque from body A to body B at ball-socket joint expressed as the following equation:

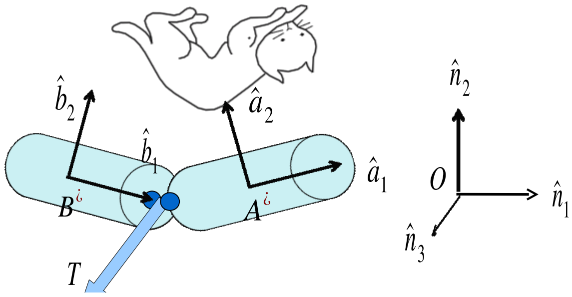

Figure 1: Model of falling cat[4]

$$\mathcal{T} = T_1\hat{a}_1 + T_2\hat{a}_2 + T_3\hat{a}_3$$

This torque was assumed to be a constant during falling. The initial conditions of falling cat were defined as released with upside-down orientation:

$q_i(0) = 0 (i = 1, 3, 4, 5, 7, 8)$
$q_2(0) = 1$
$q_6(0) = {}^{\pi}/_{12}$
$q_9(0) = -{}^{\pi}/_{6}$
$u_i(0) = 0 (i = 1, \cdots, 9)$

This means that the cat flexes 30 degree at the joint and both bodies are line symmetry with respect to the line which passes the joint. These values were determined based on the previous research[4].

Equations of motion

Based on the system described above, 18 equations of motion of the falling cat were derived using Kane's equation:

$$\dot{q_i} = f_1(u_1, \cdots, u_9, q_1, \cdots, q_9, t)\,(i = 1, \cdots, 9)$$

$$\dot{u_i} = g_1(u_1, \cdots, u_9, q_1, \cdots, q_9, t)\,(i = 1, \cdots, 9)$$

All equations are the functions of time, u_i, and q_i. Moreover, these functions include trigonometric functions and multiplication of u_i and q_i. These equations of motion are, therefore, nonlinear.

Method

The fourth-order Runge-Kutta method was applied to the equations of motion to simulate the time histories of variables.

Results

The coefficients of a given torque (T_1, T_2,T_3) were changed to know influence of the torque on the falling cat. The orientation of the upper body in space (q_4, q_5,q_6) and the relative angles between the upper body and the lower body (q_7, q_8,q_9) were used to show the falling cat movement.

Figure 2 shows the result of the orientation angles in space applied only the flexion torque at the joint. Only q_6 keep on decreasing in spite of constant q_4 and q_5. This means that the cat is falling with somersaulting.

Figure 3 shows the result of the orientation angles in space applied only the adduction and abduction torque at the joint. First, the body A rotates on the horizontal plane. Suddenly, he twists and flips about π at the same time. He repeats these two movements twice and begins to move random at last part of the graph.

Figure 4 and 5 shows the result of the orientation angles in space and relative angles between the upper and lower body applied only the twist torque at the joint respectively. From figure 4, the upper body continues twist and q_5 and q_6 oscillate with their amplitudes decreasing. Relative angles (q_8, q_9) are also oscillating in figure 5. Figure 6 and 7 show the $q_5 - q_6$ plot and $q_8 - q_9$ plot at the same condition respectively. The $q_5 - q_6$trajectory goes a fixed point and the $q_8 - q_9$trajectory draws a nearly periodic cycle. These mean that the upper body comes to be stable in orientation with spinning and the lower body draws a circle if it is seen from the upper body during falling.

Figure 8, 9, 10, 11, 12, and 13 show the results of applying two type of torques consisting of all components at the joints. In both case, the movement of the upper body shows the trajectory as that of applied only twist torque after a certain time from figure 4, 8, and 9. This means, the upper body comes to stable at a certain orientation with spinning in the space. As for the relative angels between the upper and lower body, the pattern is different in both case. In figure 10 and 12, the $q_8 - q_9$trajectory draws a nearly periodic cycle and q_7 continues to increase. However in another case, the $q_7 - q_8$trajectory draws a nearly periodic cycle and q_9 continues to decrease in figure 11 and 13.

Conclusion

A falling cat model applied a torque at the joint was developed and the movements of the falling cat were investigated in various torques. The results indicate that the upper body of the falling cat comes to a stable orientation with spinning in a certain time when the cat is applied a twist torque at its joint. Conversely, the lower body of it shows a periodic movement relative to the upper body.

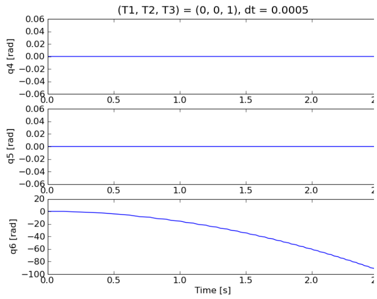

Figure 2: Time histories of q_4, q_5, q_6 angles at $\mathcal{T} = \hat{a}_3$

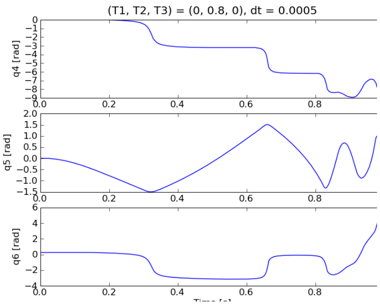

Figure 3: Time histories of q_4, q_5, q_6 angles at $\mathcal{T} = 0.8\hat{a}_2$

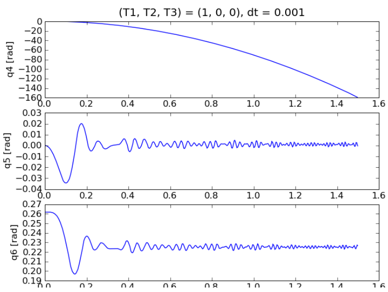

Figure 4: Time histories of q_4, q_5, q_6 angles at $\mathcal{T} = \hat{a}_1$

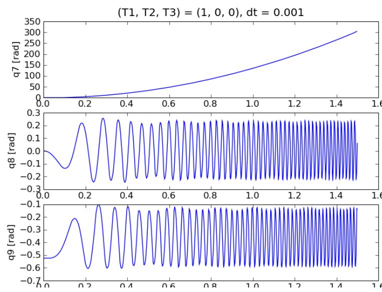

Figure 5: Time histories of q_7, q_8, q_9 angles at $\mathcal{T} = \hat{a}_1$

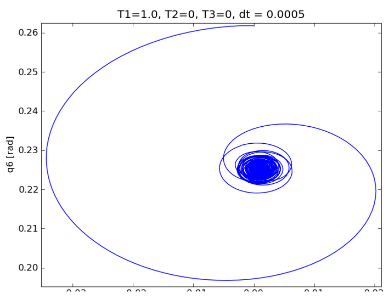

Figure 6: $q_5 - q_6$ plot at $\mathcal{T} = \hat{a}_1$

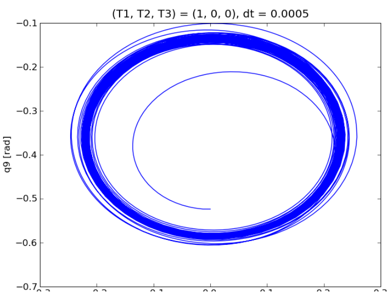

Figure 7: $q_8 - q_9$ plot at $\mathcal{T} = \hat{a}_1$

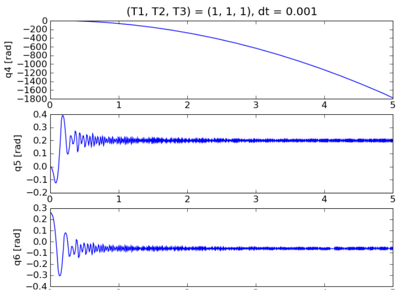

Figure 8: Time histories of q_4, q_5,q_6 angles at $\mathcal{T} = \hat{a}_1 + \hat{a}_2 + \hat{a}_3$

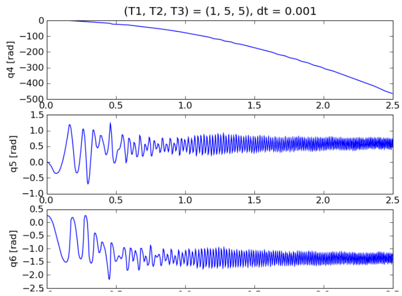

Figure 9: Time histories of q_4, q_5,q_6 angles at $\mathcal{T} = \hat{a}_1 + 5\hat{a}_2 + 5\hat{a}_3$

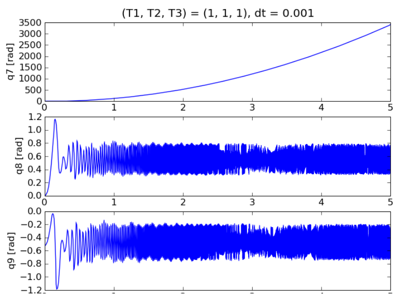

Figure 10: Time histories of q_7, q_7,q_9 angles at $\mathcal{T} = \hat{a}_1 + \hat{a}_2 + \hat{a}_3$

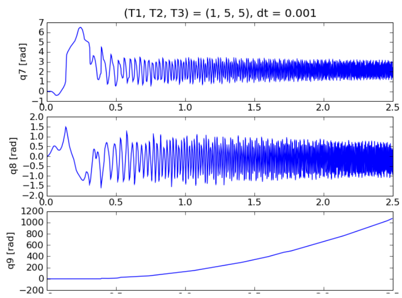

Figure 11: Time histories of q_7, q_8,q_9 angles at $\mathcal{T} = \hat{a}_1 + 5\hat{a}_2 + 5\hat{a}_3$

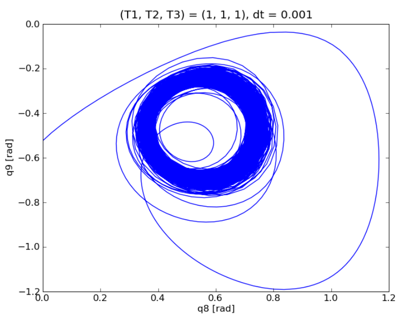

Figure 12: $q_8 - q_9$ plot at $\mathcal{T} = \hat{a}_1 + \hat{a}_2 + \hat{a}_3$

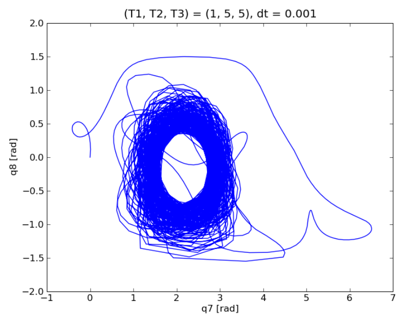

Figure 13: $q_7 - q_8$ plot at $\mathcal{T} = \hat{a}_1 + 5\hat{a}_2 + 5\hat{a}_3$

There are, however, two types of the periodic cycle movement based on the component ratio of the applied torque.

In this project, just several torque cases were observed. The bifurcation diagrams, therefore, are needed to show the movement of the falling cat precisely. Especially, the movement of the lower body relative to the upper body should be classified. For this, the python code should be rewritten in order to run less time because it takes about five minutes to run a certain case.

References

[1] http://en.wikipedia.org/wiki/Cat_righting_reflex

[2] http://en.wikipedia.org/wiki/Falling_cat_problem

[3] Kane TR & Scher MP. International Journal of Solids Structures, 5, 663-670, 1969.

[4] Xin-sheng G & Li-qun C. Applied Mathematics and Mechanics 28, 601-607, 2007.

[5] Montgomery R. Fields Institute Communications 1, 1993.