A Limit Cycle Analysis of the Simplest Passive Dynamic Walker

by:
Thomas Johnston, PhD candidate
Dept. of Biomedical Engineering
Email: trjohnston@ucdavis.edu

Abstract:
Passive dynamic walkers (PDWs) are simple dynamical systems that have been used to study the fundamentals of normal human walking. In essence, they are a collection of links and joints capable of "walking" down shallow slopes with a surprisingly human-like gait. PDWs have no active motors or control system, rather they are "powered" by gravity. With judicious choice of model parameters and initial conditions, PDWs can exhibit stable limit cycles. The goals of this project are twofold: (1) to create a general limit cycle analysis and simulation tool using Python and (2) explores some aspects of the limit cycle behavior of a PDW. In order to test the accuracy of the Python code, I recreated some of the results of by Garcia's et al., (1998) simplest passive dynamic walking model. Specifically, I verified the author's reported range of stable limit cycles. My calculations agree that this model does possess stable limit cycles for ground slopes less than about 0.015 radians.

Introduction:

Walking is an activity that most of us do on a daily basis, generally without thought or care. However, walking is an extremely complicated task and not very well understood. Much active research is devoted to answering fundamental questions regarding walking mechanics, for example:

- how much of normal walking is dictated by the classical mechanics and how much is due to neuromuscular control?
- what are some of the parameters that affect a person's self-selected walking speed?
- what factors predict when a person will switch from a walk to a run (or run to a walk)?
- is a bipedalism more or less energetically efficient that quadrupedalism?

Passive dynamic walking models are a logical means to address these questions because they are very simple and permit degrees of complexity/realism to be incrementally added. As each degree of complexity is added, the effects of the new model elements can be compared to the previous results thus illuminating the relative importance of each model element. Further, passive dynamic walkers (PDWs) provide a means to separate the skeletal muscle and nervous system, an essential requirement in order to address the significance of classical mechanics in normal walking.

Background

The proposed model is based on the work of passive walking bipeds, a concept pioneered by the works of Mochon & McMahon, (1980a, 1980b) and Tad McGeer, (1990a, 1990b) and has since been the focus of numerous studies, e.g. Goswami *et al.,* (1997), Garcia *et al.,* (1998), Kuo, (1999), Grizzle *et al.,* (2001), Hurmuzlu *et al.,* (2004), Wisse *et al.,* (2005). Mochan & McMahon noticed that electromyography (EMG) patterns of the lower extremities showed bursts of activity at the beginning and end of a step, but the EMG signal during the swing phase was largely quiescent. They also noted that the step period was reasonably close to that of a pendulum having equal inertial properties to the leg.

Theoretical and even physical models of passive walking bipeds are a collection of links and joints that represent the lower extremity of a human. With careful selection of segment masses and inertial properties, these models are able to move down gentle slopes without any control or external energy sources in a remarkably human-like fashion. If the energy lost during the foot-ground impact events is balanced by the conversion of gravitational potential energy, then a periodic gait can be achieved.

Dynamical System Description

Rigid body model description

This work begins by adopting a completely rigid body model similar to that of Garcia *et al.,* (1998). The passive walker consists of two rigid, massless legs and three particles, as shown in Figure 1. The legs have a length, l, and are connected at the "hip" by a frictionless revolute joint. The particles are idealized point masses representing the two "feet", m_f, and the combined mass of the head, arms, and trunk, m_{HAT}. The m_{HAT} is coincident with the revolute joint and is much more massive than the feet, i.e., $m_{HAT} >> m_f$. The planar model is constrained to move down a rigid ramp of slope, γ. The angle between the slope-normal and the stance leg is $\theta(t)$ and $\Phi(t)$ is the angle between the stance and swing legs. Positive directions for both $\theta(t)$ and $\Phi(t)$ are as shown in Figure 1.

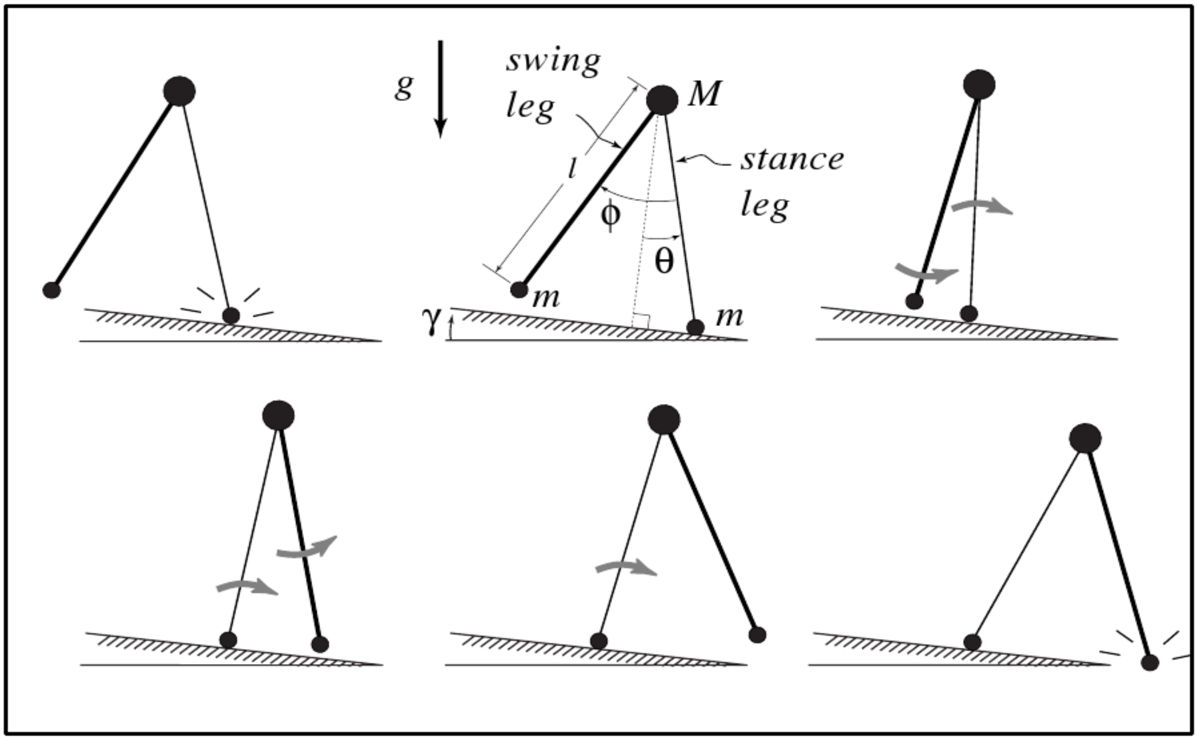

Figure 1: Kinematic description of the proposed passive walking model

Rigid body model assumptions

It is assumed that the ground is perfectly rigid and that the impact of the foot is a perfectly plastic collision with no slip and no bounce. As a consequence of this plastic impact assumption, the impact process is instantaneous and impact forces are impulsive. Further, the instantaneous impact implies that there are discontinuities within the velocity variables. Other forces during this impact period are assumed to be negligible. The contact foot is assumed to remain fixed to the ground throughout the stance phase and acts as a frictionless revolute joint. The transition of the foot from swing to stance is assumed to occur instantaneously, thus there is no double support phase. Also, the swing leg is permitted to pass through the ground-slope during mid swing. This final assumption can be justified in two ways: (1) since the feet are much less massive than the hip, the swing foot has negligible effect on the dynamics and (2) it could be implemented by using a checker-board surface [McGeer, (1990a)].

The equations of motion (EOM) for this two-degree of freedom system are derived in AUTOLEV using Kane's method and numerically integrated using a custom 4^{th} order Runge-Kutta algorithm in Python. By assuming that the mass of the hip is >> mass of the feet and non-dimensionalizing with respect to leg length and time, the EOM simplify to:

$$\ddot{\theta}(t)-\sin(\theta(t)-\gamma)=0 \tag{1}$$

$$\ddot{\theta}(t)-\ddot{\varphi}(t)+\dot{\theta}(t)^2\sin(\varphi(t))-\cos(\theta(t)-\gamma)\sin(\varphi(t))=0 \tag{2}$$

Notice that equation one describes the dynamics of the stance leg and, due to the mass of the feet being negligible, the swing foot makes no contribution to the angular momentum of the stance leg. Therefore, no terms involving phi or its time derivatives appear in equation one. Also, this system of equations has only one free parameter in these equations, the ground slope, *γ*.

By virtue of the system's geometry, the ground-foot impact event is recognized when $\Phi(t_{HS})-2\Theta(t_{HS}) = 0$, where t_{HS} is the time at heelstrike. Since this geometric condition is also satisfied at the beginning of the swing phase and as the swing leg passes in front of the stance leg, provisions in the code are made to ignore these non-interest events. It is my goal to focus on solutions of the EOM that have "real world" significance, i.e., those that will yield a stable, "period-one gait."

Methods:

The Search for Stable Periodic Solutions

A primary goal is to find initial conditions that lead to periodic gaits. A Poincaré map has proven to be a particularly useful tool in this endeavor. Geometrically speaking, a Poincaré map can be thought of as a transverse slice through an arbitrary portion of a state-space trajectory. Each time the trajectory crosses the plane defined by the transverse slice, it creates a point, $\vec{x}_n$, on the map. Since the state space trajectory depicts the evolution of the system's dynamics, there is some function that relates sequential crossings and in general may be written in the form (Hilborn, 1994),

$$\vec{x}_{n+1} = F(\vec{x}_n) \qquad (3)$$

If after *m* crossings two points on the return map are coincident, the system is said to have a period-*m* limit cycle. The discovery of a limit cycle requires finding "fixed points" of (X), i.e., a vector of initial conditions, $\vec{x}^*$, such that

$$\vec{x}^* = F(\vec{x}^*) \qquad (4)$$

McGeer, (1990b) called the function *F* in (X) a "stride map", although "step map" is perhaps a more precise term since it is mapping of the system's state at the beginning of sequential steps. Specifically, the function *F* involves the numerical integration of the EOM during single support, the detection of heel strike, the calculation of post-impact velocities, the integration of the EOM during double support, and a re-ordering of the state elements at the end of double support. In short, *F* maps the state of the system at the beginning of step *n* to the state at the beginning of step *n*+1. The re-ordering is necessary because the walker actually requires two steps to return to a similar initial configuration, e.g., beginning single support with the stance leg described by $\phi(t)$. The re-ordering ensures that the sign of each element in the state vector at the beginning of a step is the same as it was at the beginning of the previous step, which is necessary for an accurate estimate of the stride map error.

The error in the stride map is defined as,

$$g(\vec{x}) = \vec{x}_{n+1} - F(\vec{x}_n) \qquad (5)$$

In general, it is impossible to blindly guess a set of initial states that is a fixed point, but a Newton-Raphson (NR) algorithm often leads to a root of $g(\vec{x})$. The NR algorithm relies on approximating the error function as a Taylor series expansion about the current state, $\vec{x}$,

$$g_i(\vec{x} + \delta\vec{x}) = g_i(\vec{x}) + \sum_{j=1}^{N} J_{ij}(\vec{x}) \cdot \delta x_j + O(\delta\vec{x}^2) \qquad i, j = 1, 2, \ldots, N$$

where *N* is the number of elements in the state vector and $J_{ij}(\vec{x})$ is the Jacobian of *g* evaluated at $\vec{x}$, i.e.,

$$J_{ij}(\vec{x}^*) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

and solving for $\delta\vec{x}$. Neglecting the higher order terms, the change to the initial guess that will drive the system towards the fixed point is

$$\delta\vec{x} = [J_{ij}(\vec{x})]^{-1} \cdot (g(\vec{x} + \delta\vec{x}) - g(\vec{x})) \quad (6)$$

Thus, the next iteration of the stride map is

$$\vec{x}_{new} = \vec{x}_{old} + \delta\vec{x} \quad (7)$$

Notice that the calculation of $\delta\vec{x}$ is a computationally expensive task. In total, n+1 iterations of the stride map are required, once using the nominal state vector, $\vec{x}$, and n more times for the numerical approximation of the Jacobian. It is also worth noting that the i^{th} column of the Jacobian represents the sensitivity of the error function to a small change (perturbation) in the i^{th} element of the state vector while all other elements of $\vec{x}$ are held constant.

Once a limit cycle has been found, its linear stability is checked with a similar procedure. We perform a Taylor series expansion of F about the fixed point, retaining only the linear terms. If the eigenvalues of the Jacobian of $F(\vec{x}^*)$ are all strictly less than one, then sufficiently small perturbations will converge to the fixed point and the limit cycle is asymptotically stable [Goswami *et al.,* (1998)].

Disclaimer:
I worked hard to get my Newton-Raphson algorithm working perfectly but ran out of time. I tried to be careful about making a standard input-output form passing arguments between functions; however, I wasn't careful enough. In the end I used scipy's *fsolve* function.

Results:

The swing phase dynamics are as expected and, in fact, seem to match that of the authors' results perfectly. Both leg angles are initially positive (note that the initial swing leg angle is equal to twice that of the stance leg angle – a configuration that would have both feet on the ground) and both monotonically decrease until phi equals 2*theta. The swing phase dynamics do seem to be largely pendular in nature, agreeing with the observations of Mochon & McMahon, (1980a, 1980b). The non-dimensional time at which the foot strikes the ground is 3.867 (time is non-dimensionalized by multiplying *t* by sqrt(g / L)).

In terms of the stance leg angle corresponding to a fixed point, my code also matched the predictions of Garcia et al., (1998). The authors found two fixed points over a range of ground slopes between 0 and 0.045 radians. One fixed point had a slightly longer step time (period) and was thus called the "long period" solution, while the other was referred to as the short period solution (Figure 4 in Garcia's paper). The major results reported by Garcia et al., (1998) were that: (1) the leg angles of both the long and short period solutions scaled as proportional to the ground slope angle to the (1/3) power, (2) the long period solution was stable up until ground slopes of about 0.015 radians, and (3) the short period solutions were always unstable. Again, I was able to recreate Garcia's results perfectly. One minor detail

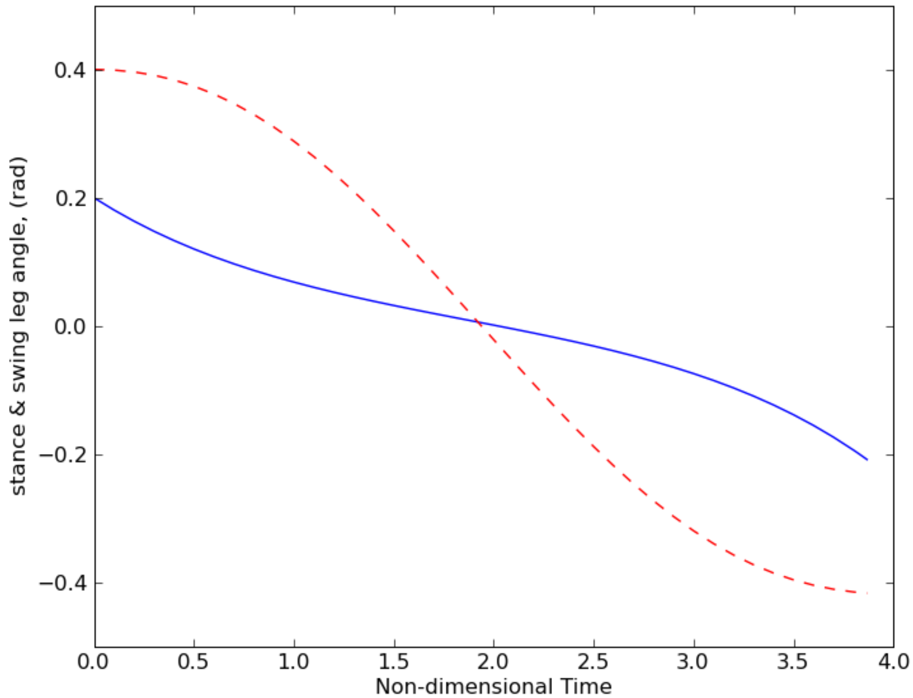

Figure 2: A comparison of my results with those of Garcia et al., (1998). The stance leg (solid line) and swing leg (broken line) seem to match the author's Figure 2 perfectly. Here, gamma = 0.009 radians.

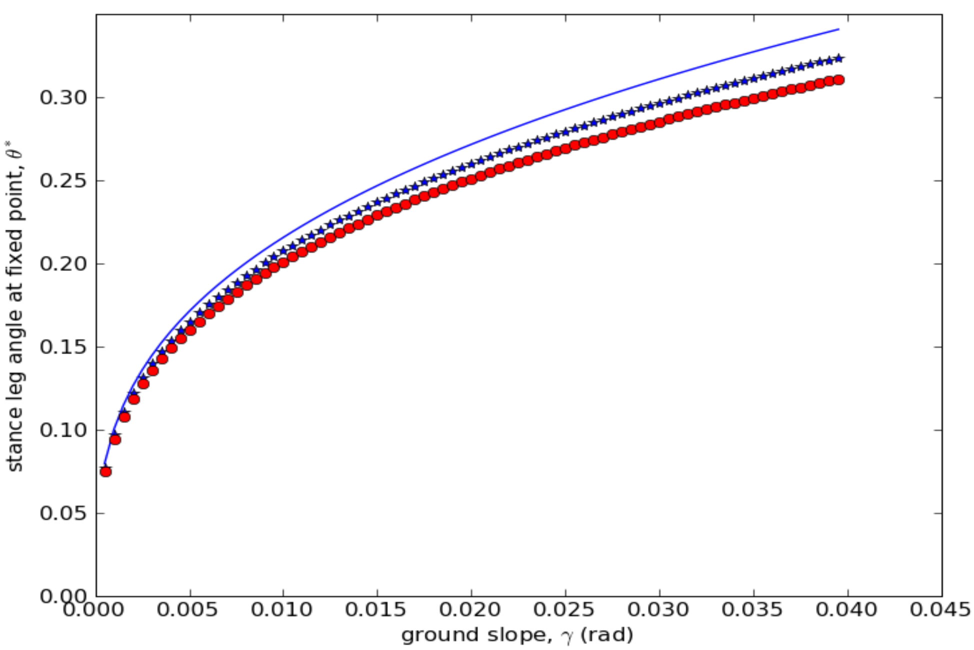

Figure 3: A comparison of my results with those of Garcia et al., (1998). The (*) show my numerical calculations of the stance leg angle (theta) corresponding to the fixed point at a given ground slope (gamma). My results seem to agree with the author's assertion that the stance leg angle scales approximately to gamma to the 1/3 power.

to notice when comparing my Figure 3 to Garcia's Figure 3 is that I plot his proposed trend line, gamma$^{(1/3)}$, shown as the solid line, while Garcia plots an analytical approximation to the fixed point. Clearly, the proposed trend line is a reasonable description of the affects of ground slope on the stance leg angle.

Finally, I compared the eigenvalues of the fixed points for both the long and short period solutions. My results (Figure 4) are in agreement with Garcia (Figure 4). The short period solutions are always unstable. The long period solution is stable until approximately gamma = 0.015 radians, at this point the magnitude of the eigenvalues is greater than 1.

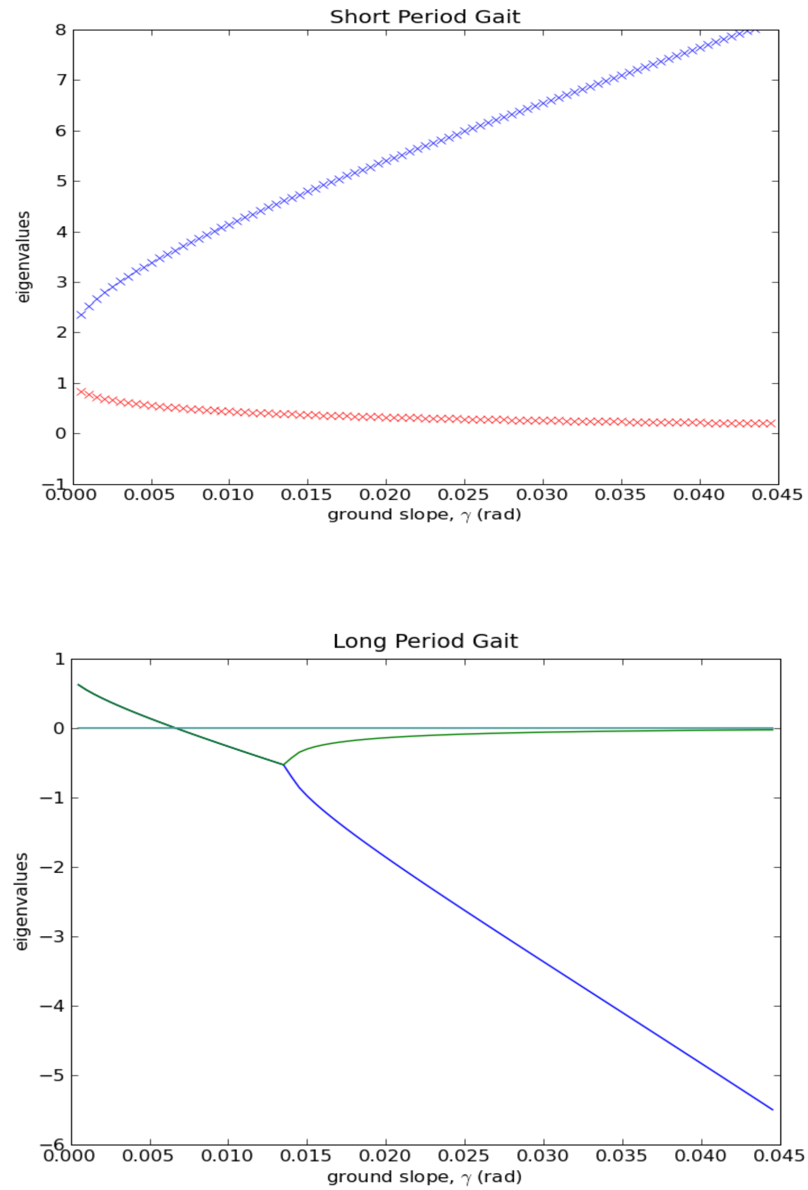

Figure 4: A comparison of the stability results

The Code:
The code is contained within two files: Walker.py (the main file) and WalkerDefs.py. No attempt was made to program this in an object-oriented fashion.

Conclusion:

This project was an attempt to build a analysis and simulation tool of a dynamical system using Python. The dynamical system chosen as a representative model is a challenging one because of its hybrid nature, i.e., it exhibits continuous time dynamics that are interrupted by discrete events. It is not a trivial task to detect and terminate an integration routine at a time that is not known *a priori*. This tool also demonstrates a general procedure for finding fixed points of a limit cycle and evaluating their stability. While still very much in a beta stage of development, the functionality is largely there as it showed general agreement with the published results of the simplest walking model.

Bibliography:

Garcia M, Chatterjee A, Ruina A, Coleman M. The simplest walking model: Stability, complexity, and scaling. ASME Journal of Biomechanical Engineering 1998;120(2):281-288.

Goswami A, Espiau B, Keramane A. Limit cycles in a passive compass gait biped and passivity-mimicking control laws. *Autonomous Robots* 1997;4(3):273-286.

Goswami A, Thuilot B, Espiau B. A study of the passive gait of a compass-like biped robot: Symmetry and chaos. *Inter J Robotics Research* 1998;17(12):1282-1301.

Grizzle JW, Abba G, Piestan F. Asymptotically stable walking for biped robots: Analysis via systems with impulse effects. *IEEE Trans on Automatic Control* 2001;46(1):51-64.

Hilborn RC. Chaos and Nonlinear Dynamics: An Introduction for Scientists and Engineers. Oxford University Press, New York, 1994.

Hurmuzlu Y, Génot F, Brogliato B. Modeling, stability and control of biped robots – a general framework. *Automatica* 2004;40:1647-1664.

Kuo AD. Stabilization of lateral motion in passive dynamic walking. *Intl J Robot Res* 1999;18(9):917-930.

McGeer T. Passive dynamic walking. *Int J Robot Res* 1990a;9(2):62-82.

McGeer T. Passive walking with knees. *Proceedings of the IEEE Conference on Robotics and Automation* 1990b;2:1640-1645.

Mochon S, McMahon TA. Ballistic walking. *J Biomech* 1980a;13(1):49-57.

Mochon S, McMahon TA. Ballistic walking – An improved model. *Mathematical Biosciences* 1980b;52(3-4):241-260.

Wisse M, Schwab AL, van der Linde RQ, et al. How to keep from falling forward: Elementary swing leg action for passive dynamic walkers. *IEEE Transactions on Robotics* 2005;21(3):393-401.