A Glance at the Standard Map

By Ryan Tobin

PHY 150/250

Chaos in Hamiltonian Systems

Hamiltonian Systems Overview

- Dynamical systems that can be written in the form:

$$\dot{p} = -\frac{\partial H}{\partial q}\ , \quad \dot{q} = -\frac{\partial H}{\partial p}\ .$$

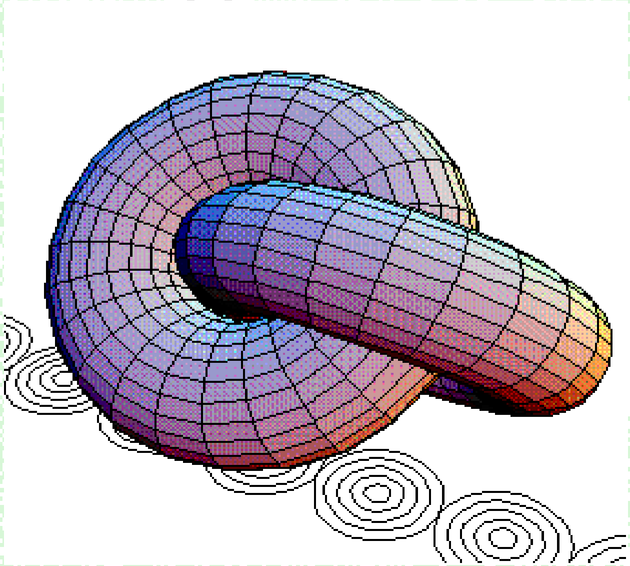
- In Physics, *H* often represents the total energy of system (kinetic + potential)
- approach is useful for analyzing conservative systems
- Hopefully the system is completely integrable

- Phase-space volume is conserved
  - A result of conservation properties of the system

- Integrable systems can be represented by a set of action variables corresponding to an invariant (KAM) torus.

Example: 2-body problem

- Conservation of energy/angular momentum
- Depending on energy the trajectories can be described to lie on the surface of torus
  - Periodic or quasi-periodic



What about non-integrable systems?

- Screwed?

Non-integrable Hamiltonian Systems

- Can gain theoretical results for the behavior of such systems by studying area-preserving iterated maps
  - These type of maps model a Poincaré section for Hamiltonian systems

Chirikov Standard Map: Intro

Defined by:

$$\theta_{n+1} = \theta_n + p_{n+1} \equiv f \text{ , modulo 1}$$

$$p_{n+1} = p_n + K/(2\pi)\sin(2\pi\theta_n) \equiv g \text{ , modulo 1}$$

As I have written it to fit on a (0,1) by (0,1) map.

Intro cont'd

- Describes the motion of a kicked rotator
- Where θ_n and p_n determines the angular position and momentum respectively
- The constant K is the intensity of the "kicks"
  - Measures how nonlinear the system is
- Many physical systems and maps can be reduced (locally) to the standard map
  - So widely applicable, thus its name.

Intro cont'd*2

- The standard map is fundamentally very interesting :
  - a very simple depiction of chaos in an Hamiltonian conservative system
- The standard map can be viewed as a two-dimensional slice or Poincaré plane
  - Trajectories of the system (residing on surface of tori), produce elliptical/hyperbolic-like curves in the 2-d representation

Intro*3

- Conservation of volume in the phase-space translates to area-conservation in the 2-d section
- Can be easily iterated numerically to exhibit features that resemble typical conservative continuous systems

Jacobian

$$J = \begin{vmatrix} \frac{\partial f}{\partial \theta_n} & \frac{\partial f}{\partial p_n} \\ \frac{\partial g}{\partial \theta_n} & \frac{\partial g}{\partial p_n} \end{vmatrix} = \begin{vmatrix} (1 + K\cos(2\pi\theta_n)) & 1 \\ K\cos(2\pi\theta_n) & 1 \end{vmatrix} = 1$$

Thus, the map is conservative (area preserving).

Analysis

- With fixed *p*, we can write the function of angle:

$$\theta_{n+1} = \theta_n + F(p)$$

For rational *F*(*p*), the mapping produces periodic trajectories. Irrational *F*(*p*) produces quasi-periodic trajectories.

- For K=0, the standard map becomes

$$p_{n+1} = p_n \qquad , (mod\ 2\pi)$$

$$\theta_{n+1} = \theta_n + p_{n+1} \qquad , (mod\ 2\pi)$$

The solution is simply

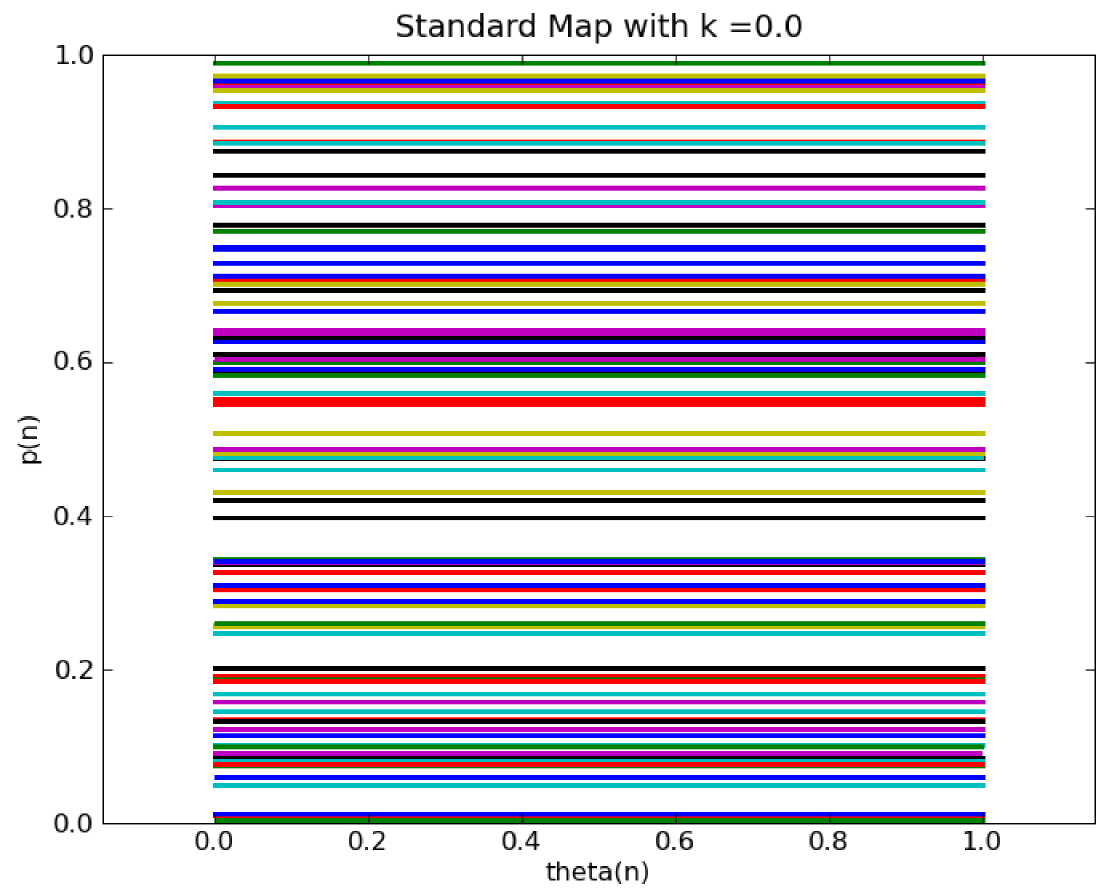
$$\theta_n = \theta_0 + n p_0 \qquad , (mod\ 2\pi)$$

Where $p_n = p_0$.

- For rational p_0 ($p_0 = M/N$; *M,N integers),* the trajectories on the torus are periodic with a period of N iterates.
- For irrational p_0, then $\theta_n \neq \theta_0$ for all n. These quasi-periodic trajectories are referred to as KAM tori

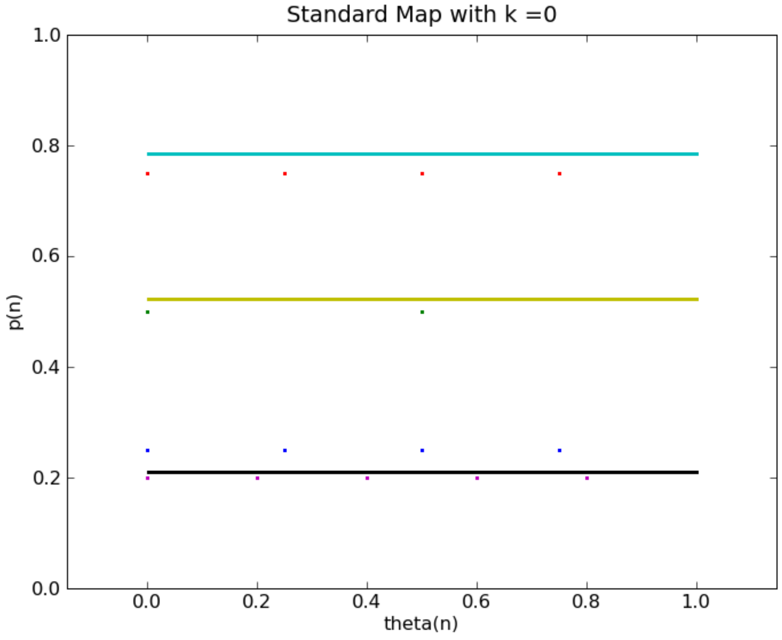
So for a K, the initial conditions determine whether the trajectories are periodic or quasiperiodic. We can refer to this rationality/irrationality as the winding numbers.

Some Plots

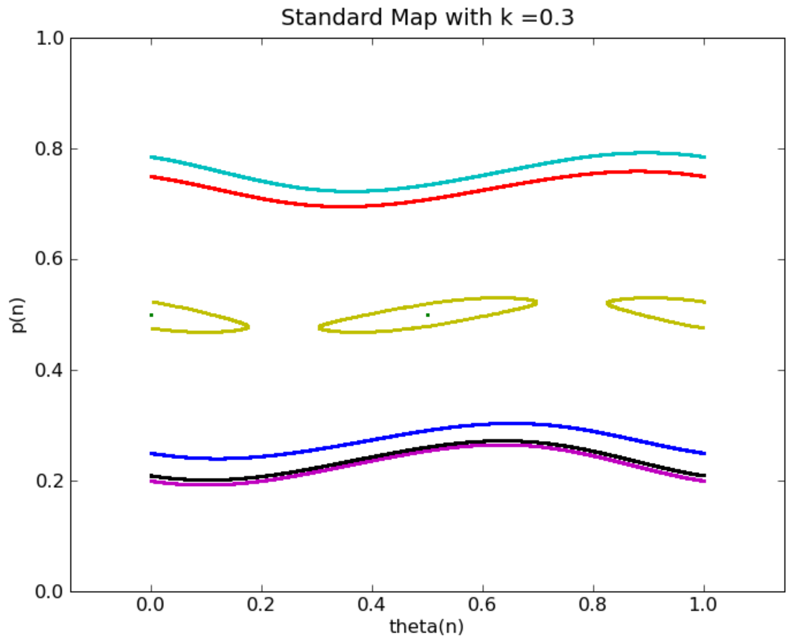


Standard Map with k =0.0
p(n)
theta(n)
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

More Plots

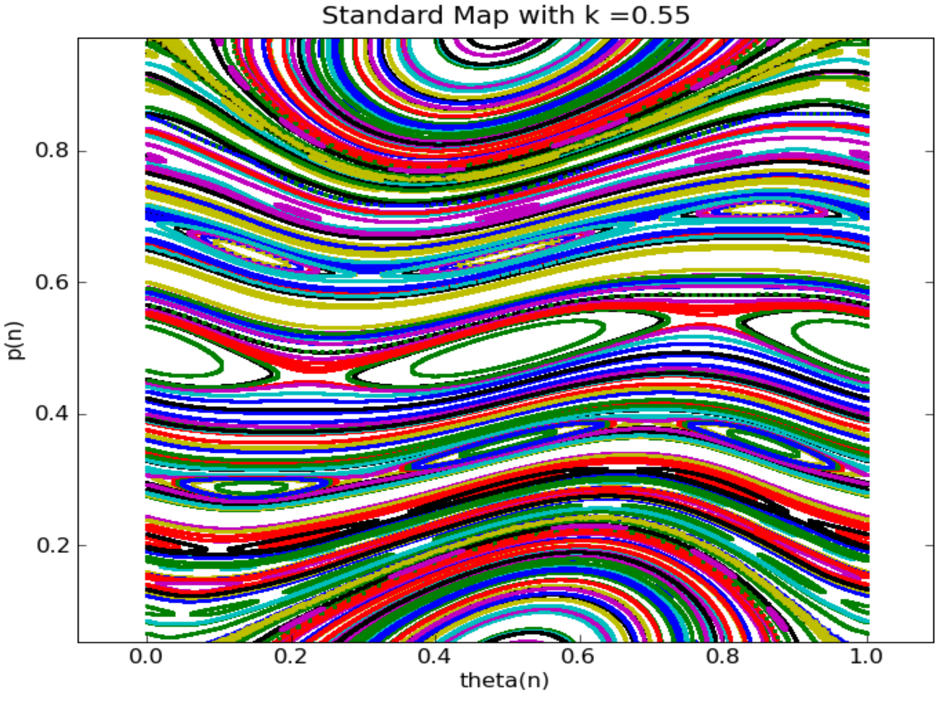


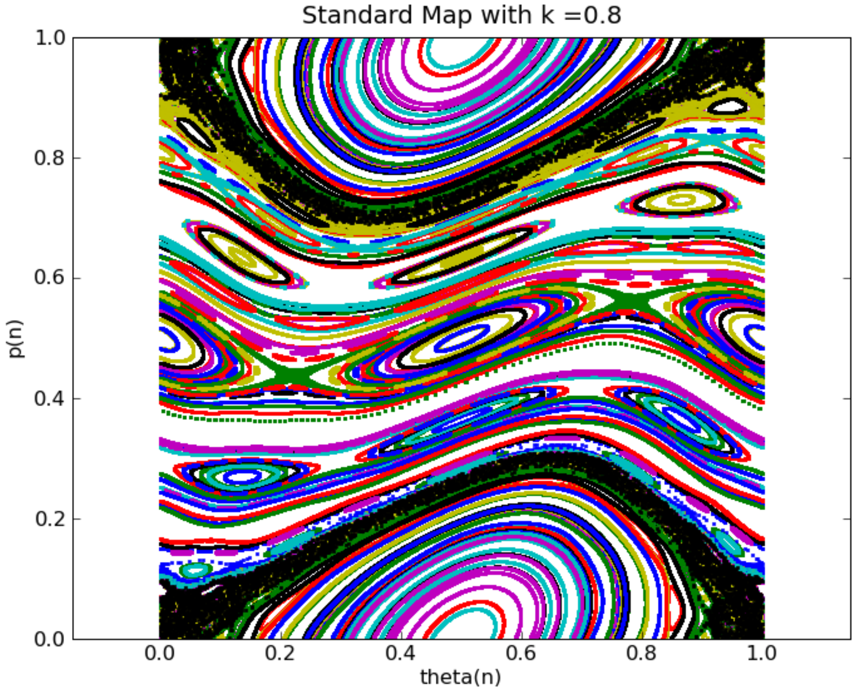
$P_0 = \pi/4, 3/4, \pi/6, ½, ¼, \pi/15, 1/5$



Nonlinear Resonances

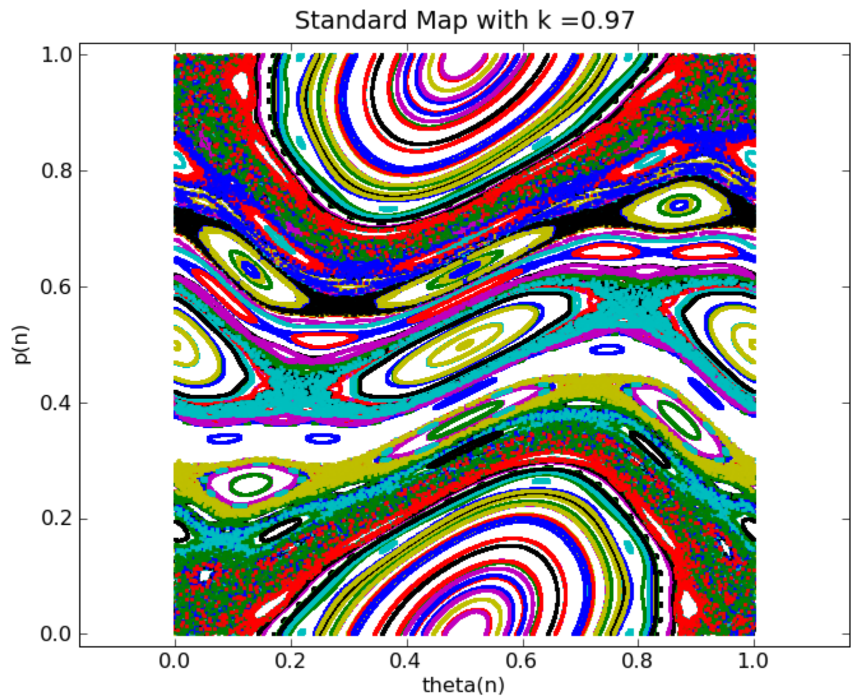
- If the frequencies of motion of different dimensions are related rationally, then periodic orbits in phase space create a nonlinear resonance.
- Near trajectories encircle these periodic fixed points forming islands-the size grows with K, and location of the resonance changes with K as well.





Nonlinear Resonances: Chaos

When resonances overlap, nearby trajectories become "confused" causing the destruction of KAM tori and localized regions of chaotic trajectories around the broken torus



Fixed points and stability

We can determine stability by the trace of the Jacobian

$$Tr(J) = 2+K\cos(2\pi\,\theta)$$

<u>Fixed Point Stability</u>

Stable: $|Tr(J)| < 2$; fixed point is a center

Unstable: $|Tr(J)| > 2$; fixed point is a saddle point

Period-1:

we can find period-one fixed points be solving the following:

$$p = p + K\sin(\theta) \quad , (mod\ 2\pi)$$

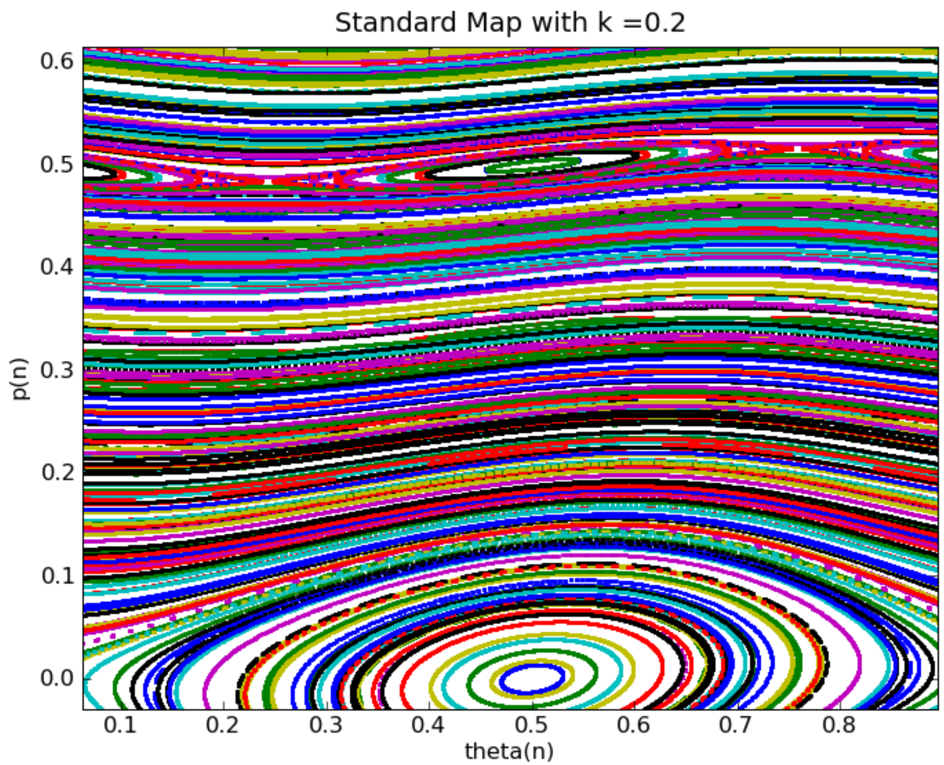
$$\theta = \theta + p \quad , (mod\ 2\pi)$$

The fixed points are θ=0,.5 and *p*=0. Directly analogous to the simple pendulum.

For small K looks just like the phase space of the simple pendulum.

Since det*(J)* = 1, the eigenvalues, and thus the stability, of the fixed points are completely determined by *Tr*(*J*).

From *Tr*(*J*), we find that (0,0) is unstable for *K* > 0 and (.5,0) stable for 0 < K < 4 .



Stable/Unstable Manifolds

- The unstable and stable sets, W^U and W^s, of an invariant set are defined as

$$W_\Lambda^S \equiv \{(x,y): f^n(x,y) \to \lambda \; as \; n \to \infty\}$$

$$W_\Lambda^U \equiv \{(x,y): f^n(x,y) \to \lambda \; as \; n \to -\infty\}$$

Where n represents the nth iterate of the standard mapping function *f*.

When represents a saddle point, the eigenvectors point in the directions of the submanifolds.

$$\lambda_{1,2} = ½\left[Tr(J) \pm \sqrt{Tr(J)^2 - 4\Delta}\right] = ½\left[2 + K\cos(2\pi\,\theta) \pm \sqrt{K^2\cos^2(2\pi\,\theta) + 4K\cos(2\pi\,\theta)}\right]$$

With corresponding eigenvectors $\boldsymbol{V}_{1,2}$.

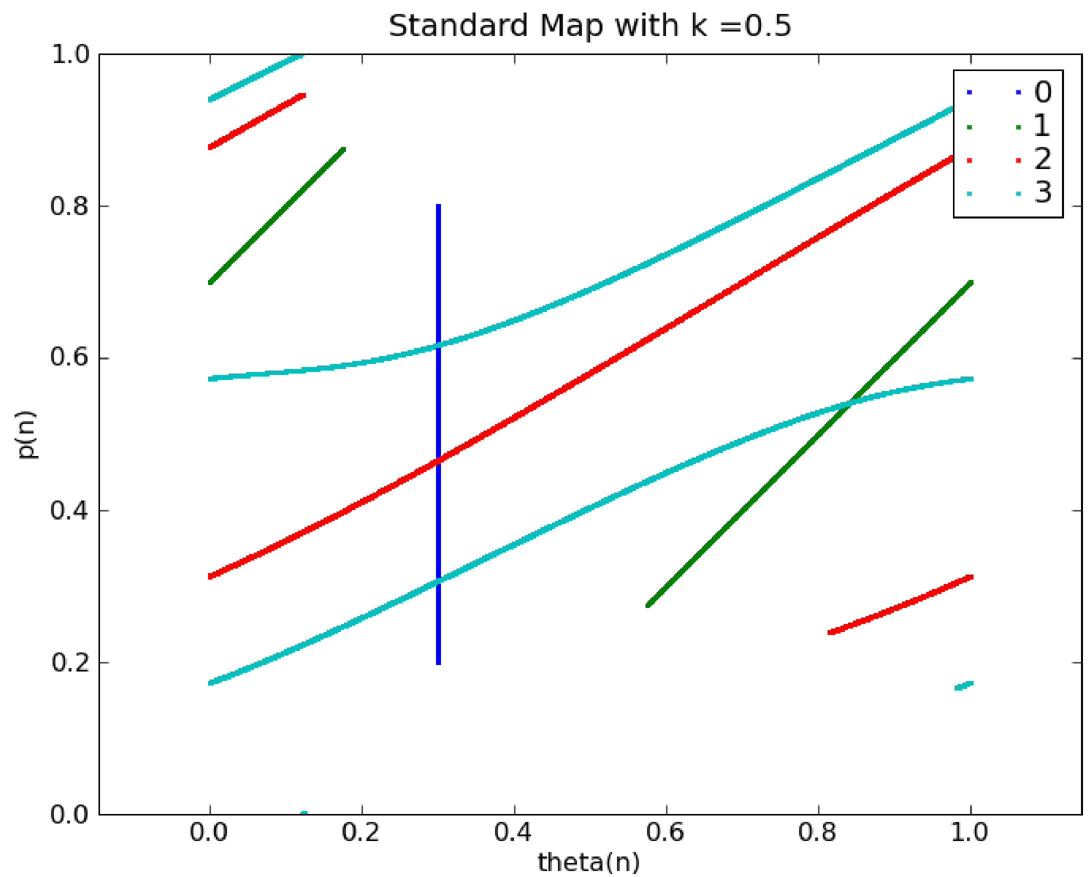
$$v_{1,2} = \left(\frac{1}{-\frac{K}{2}\cos(2\pi\,\theta) \mp ½\sqrt{K^2\cos^2(2\pi\,\theta) + 4K\cos(2\pi\,\theta)}}\right)$$

For K>0, $\boldsymbol{V_1}$ corresponds to the unstable manifold direction, and the opposite for $\boldsymbol{V_2}$.

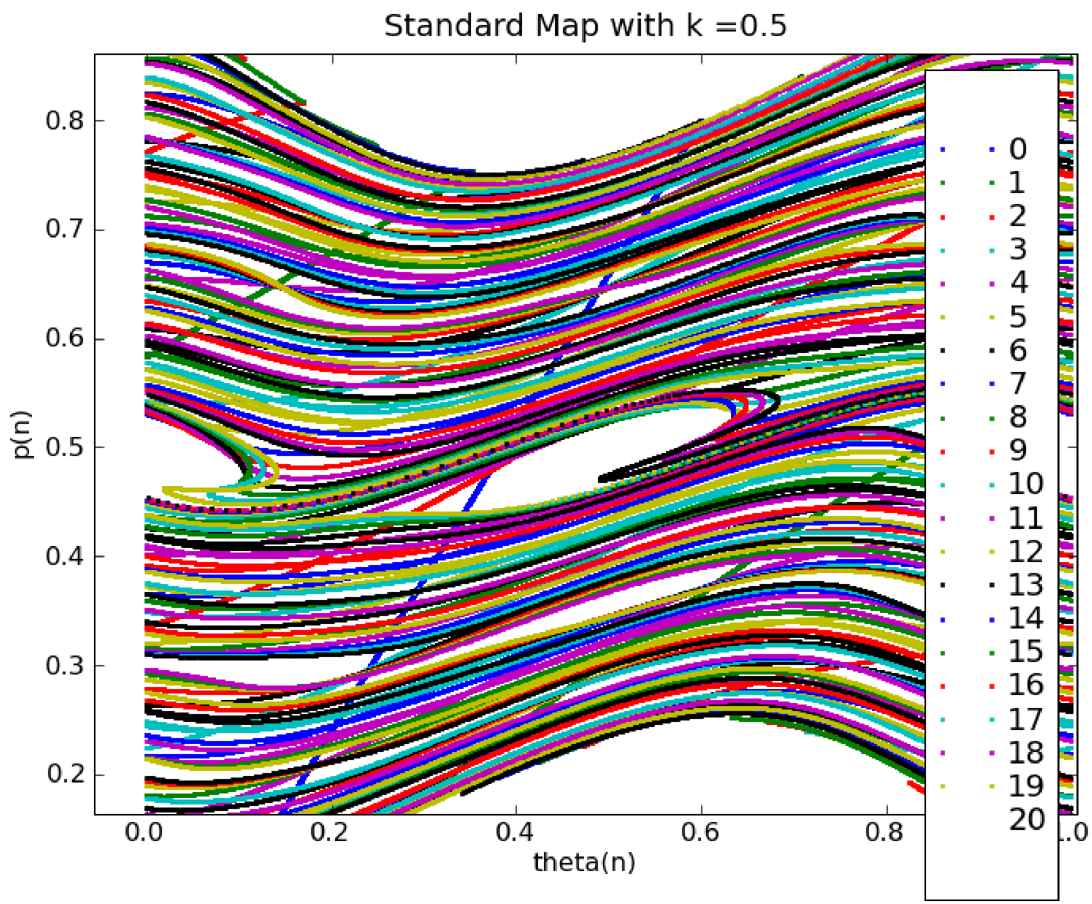
For example, the fixed point (0.5,0) has eigenvalues

$\lambda_{1,2} = ½\left[2 + K \pm \sqrt{K^2 + 4K}\right]$ and eigenvectors $v_{1,2} = \begin{pmatrix} 1 \\ 1 - \lambda_{2,1} \end{pmatrix}$.

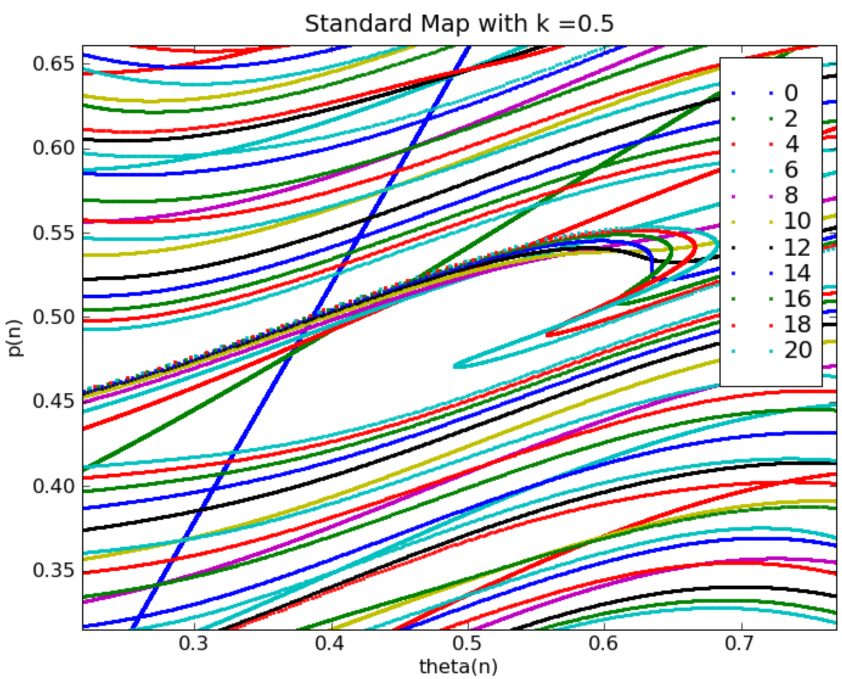
Twisties

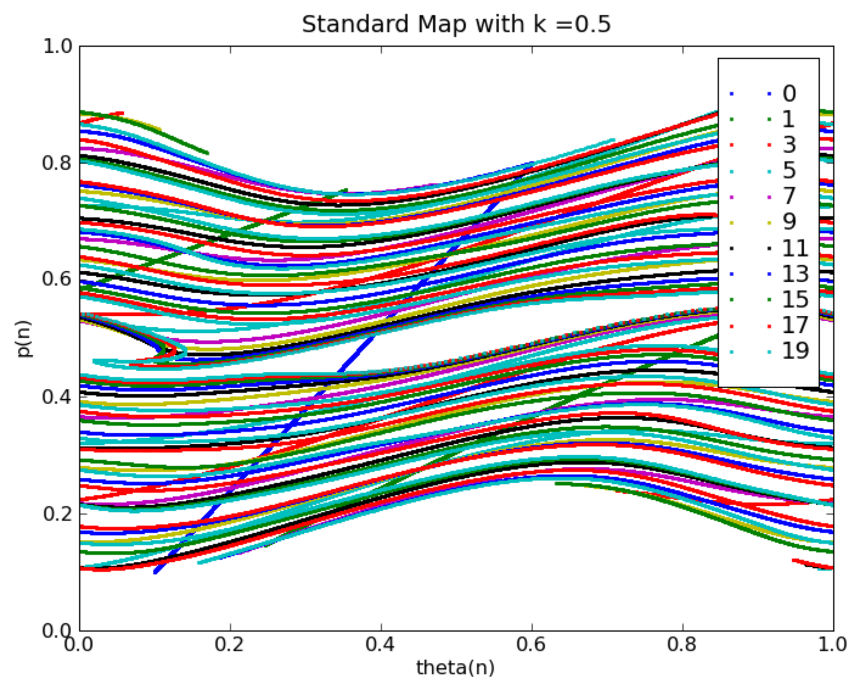

Standard Map with k =0.5
p(n)
theta(n)
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3

Twisties 2


Standard Map with k =0.5
p(n)
0.8
0.7
0.6
0.5
0.4
0.3
0.2
theta(n)
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20

Twsties 3

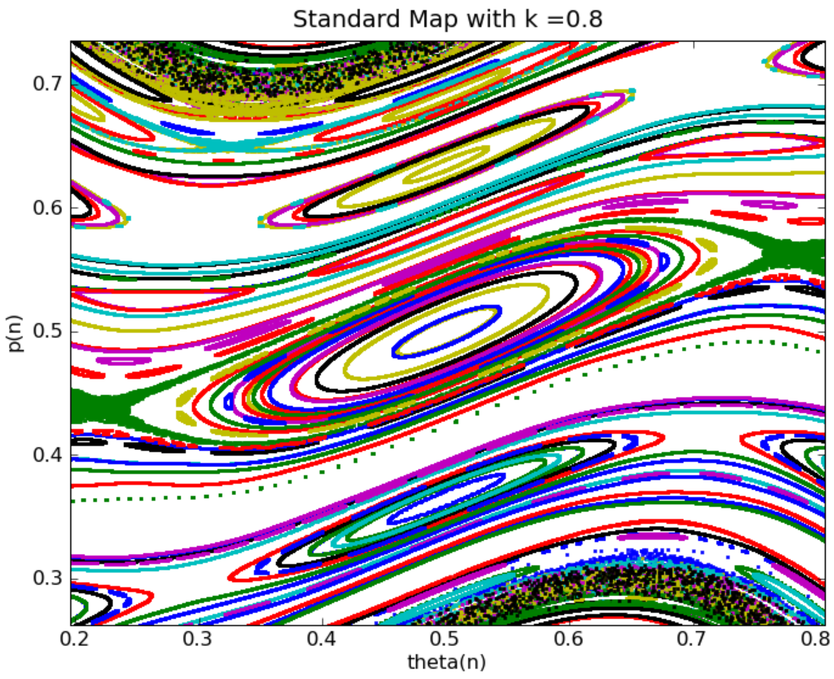

Standard Map with k =0.5
0.65
0.60
0.55
0.50
0.45
0.40
0.35
p(n)
0.3
0.4
0.5
0.6
0.7
theta(n)
0
2
4
6
8
10
12
14
16
18
20

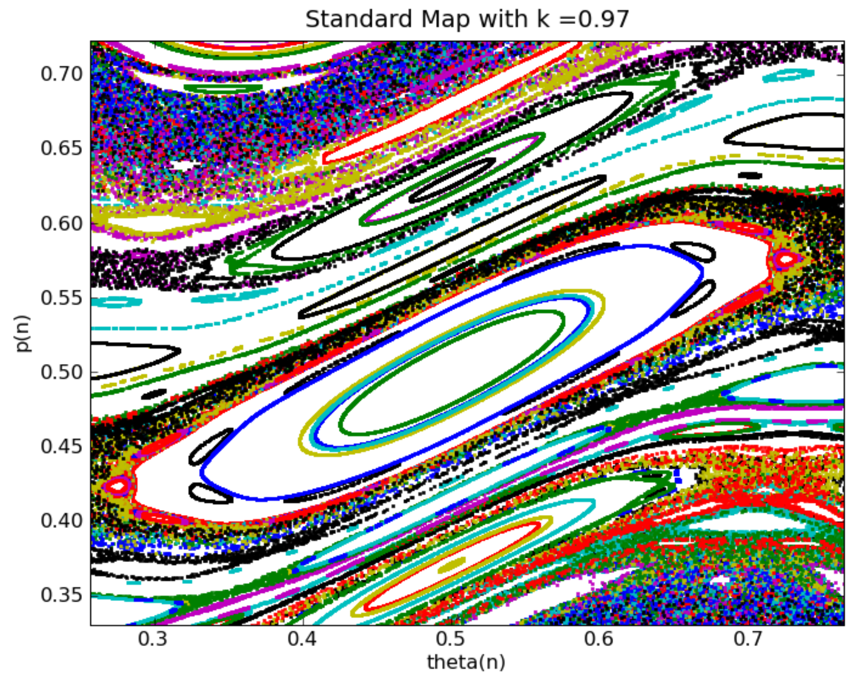

Standard Map with k =0.5
1.0
0.8
0.6
0.4
0.2
0.0
p(n)
0.0
0.2
0.4
0.6
0.8
1.0
theta(n)
0
1
3
5
7
9
11
13
15
17
19

Kolmogorov–Arnold–Moser theorem

- Describes quasi-periodic motions resulting from small perturbations.
- Motion of an integrable system is confined to an invariant torus.
- Under small perturbations, some invariant tori are deformed and remain, while others are destroyed.
- Survivors have “sufficiently irrational” frequencies (the non-resonance condition).
- The KAM theorem shows to what extent can motions remain on invariant tori with perturbations

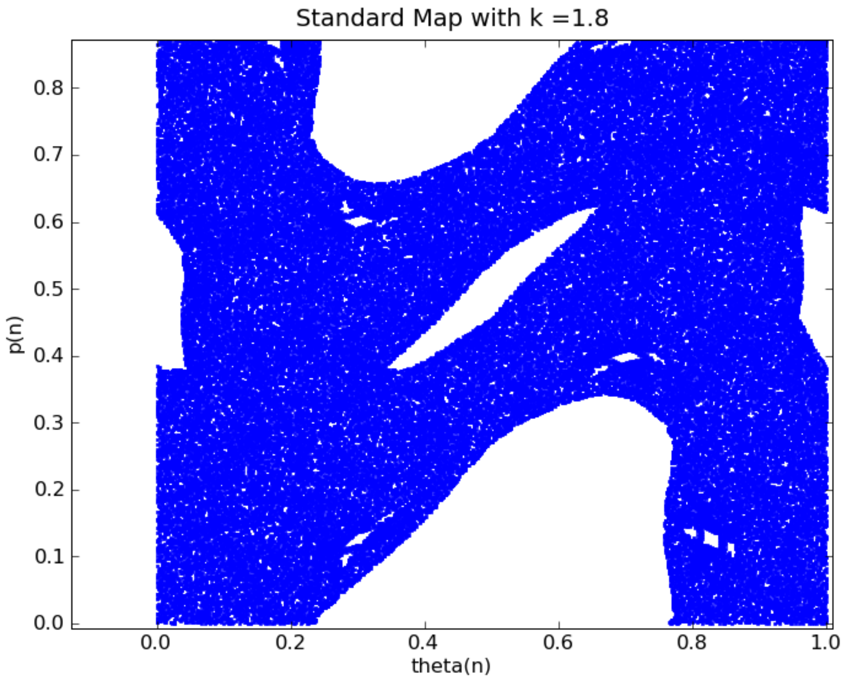
KAM tori destruction

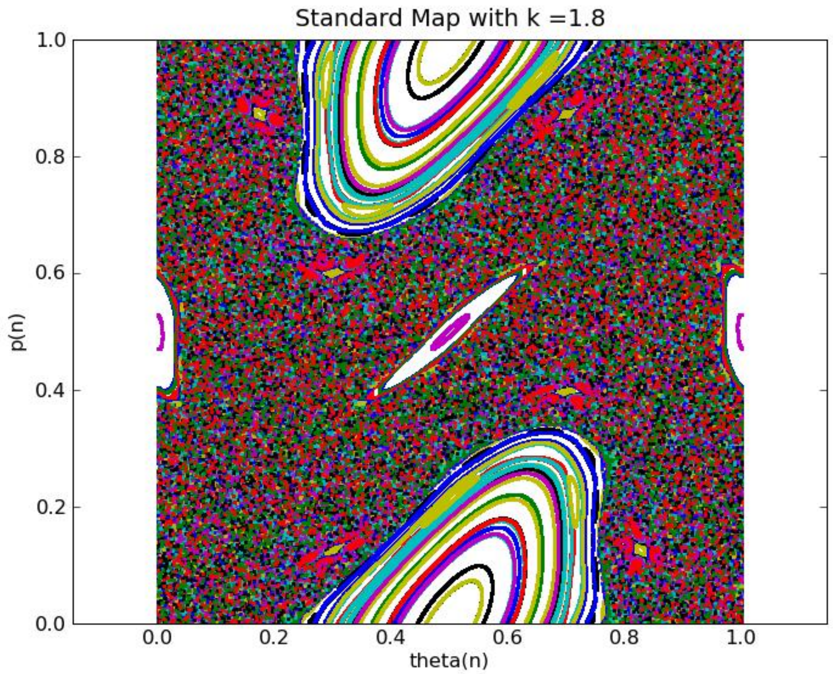




Nonlinear Resonances: Global Chaos

"Chaotic Sea"





A chaotic trajectory

How do KAM die?

- It is thought that KAM tori that live nearest a nonlinear resonance will die first
  - Thus, irrational winding numbers that are “nearest” a rational number will tend to die first.
  - Winding numbers that are furthest away tend to die later
- Irrational winding numbers whose rational expansion converge slowly to a rational number will be more durable
  - The slowest converging irrational number is the golden mean ($\frac{1}{2}+\frac{\sqrt{5}}{2}$≈1.61).
  - This last KAM torus is called the Golden Torus corresponding to a K = .971635. At this winding number, the torus is the last to break apart.

Further Considerations

- Mapping of Lyapunov exponents
- Stable/unstable manifolds and their tanglement
- Numerical predictions of Golden Torus

THE END

Questions? Comments?