Hénon–Heiles Hamiltonian – Chaos In 2-D

Modeling Chaos & Complexity – 2008

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Abstract

Chaos in two degrees of freedom (4 coordinates), demonstrated by using the Hénon–Heiles Hamiltonian

INTRODUCTION

- In 1964, Michael Hénon and Carl Heiles were investigating the motion of a star around a galactic center
- Hamiltonian equations – typical physical problem
- We have energy conservation, so we do not want the phase space to contract
- Interesting dynamic with chaotic and non-chaotic regions

Introduction continued

- The Hamiltonian governing this motion will have three degrees of freedom (6 coordinates in phase space)
- Hénon and Heiles wanted to simplify the problem
  - Energy conservation
  - Angular momentum conservation (due to the cylindrical symmetry)
  - Observation, motion is confined into two dimensions
- Chose to use a near elliptic motion

The Hamiltonian

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{1}{2}kr^2 + \frac{1}{3}\lambda r^3 \sin(3\theta)$$

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k(x^2 + y^2) + \lambda(x^2 y - \frac{1}{3}y^3)$$

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

Basic Analysis

$$\dot{x} = p_x$$

$$\dot{y} = p_y$$

$$\dot{p}_x = -x - 2xy$$

$$\dot{p}_y = -y - x^2 - y^2$$

Fixed Points

$$x = -2xy$$

$$x = 0, y \neq 0$$

$$or$$

$$x \neq 0, y = -\tfrac{1}{2}, \quad \text{x is a func of E}$$

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1-2y & -2x & 0 & 0 \\ -2x & -1-2y & 0 & 0 \end{pmatrix}$$

$$Det(J) = 1 - 4x^2 + 4y + 4y^2$$
$$Tr(J) = 0$$

$$\lambda_{1,2} = \pm\sqrt{-1-2x-2y}$$
$$\lambda_{3,4} = \pm\sqrt{-1+2x-2y}$$

Sums to zero, can be only real or only imaginary

$$Det(J) = 1 + 4y + 4y^2,\ with\, x = 0 \Rightarrow 2E = y^2 - \frac{2}{3}y^3$$

$$Det(J) = -4x^2,\ with\, y = -\tfrac{1}{2} \Rightarrow E = 1/6$$

Using the energy conservation

For x = 0 Det(J) = 1 only for y = 0

For y = -1/2 |Det(J)|=1 for x = 0.25

Alarming since I expected the Det(J) to be 1 all the time

The Potential

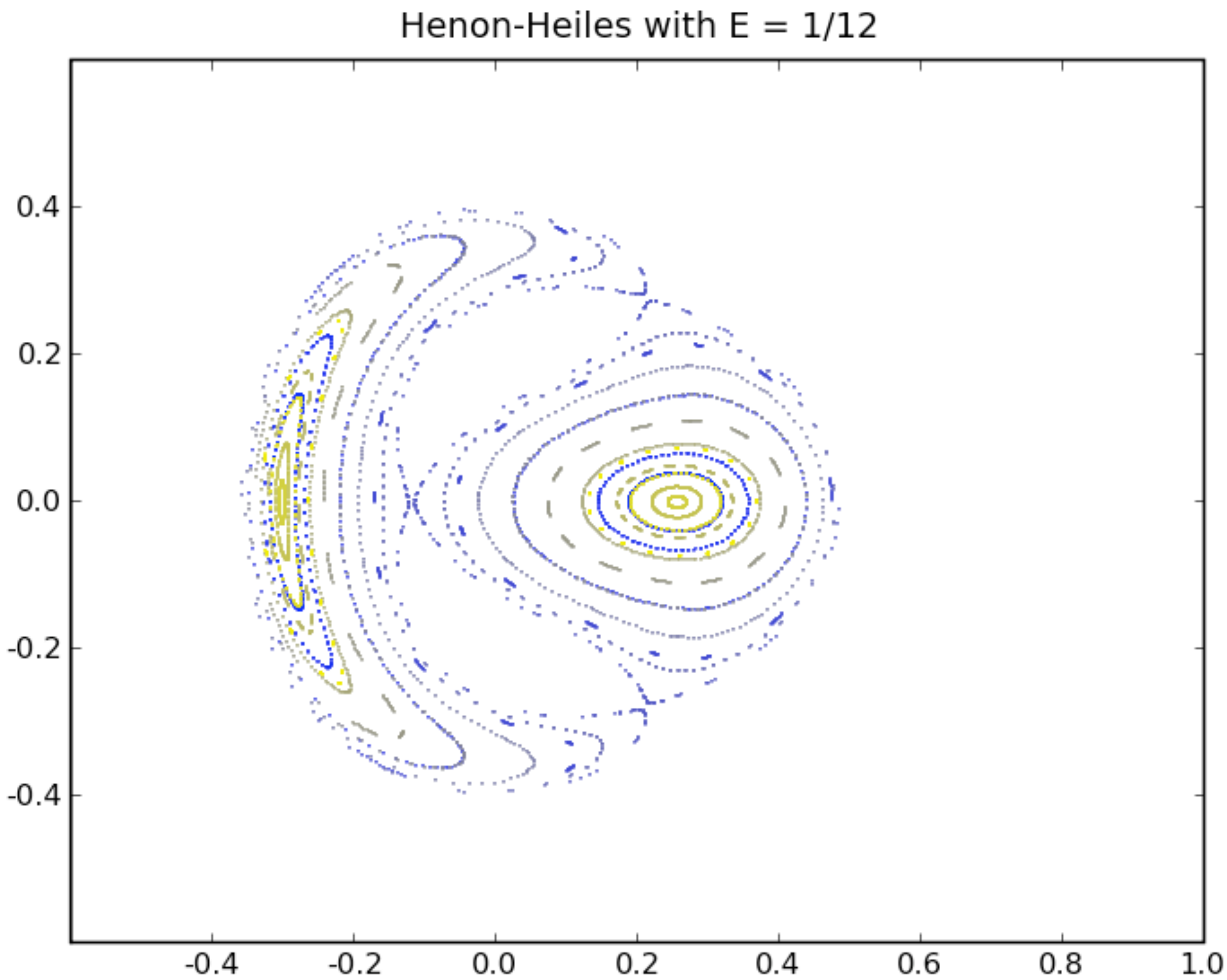
$$V(x,y)=\frac{1}{2}(x^2+y^2+2x^2y-\frac{1}{3}y^3)$$

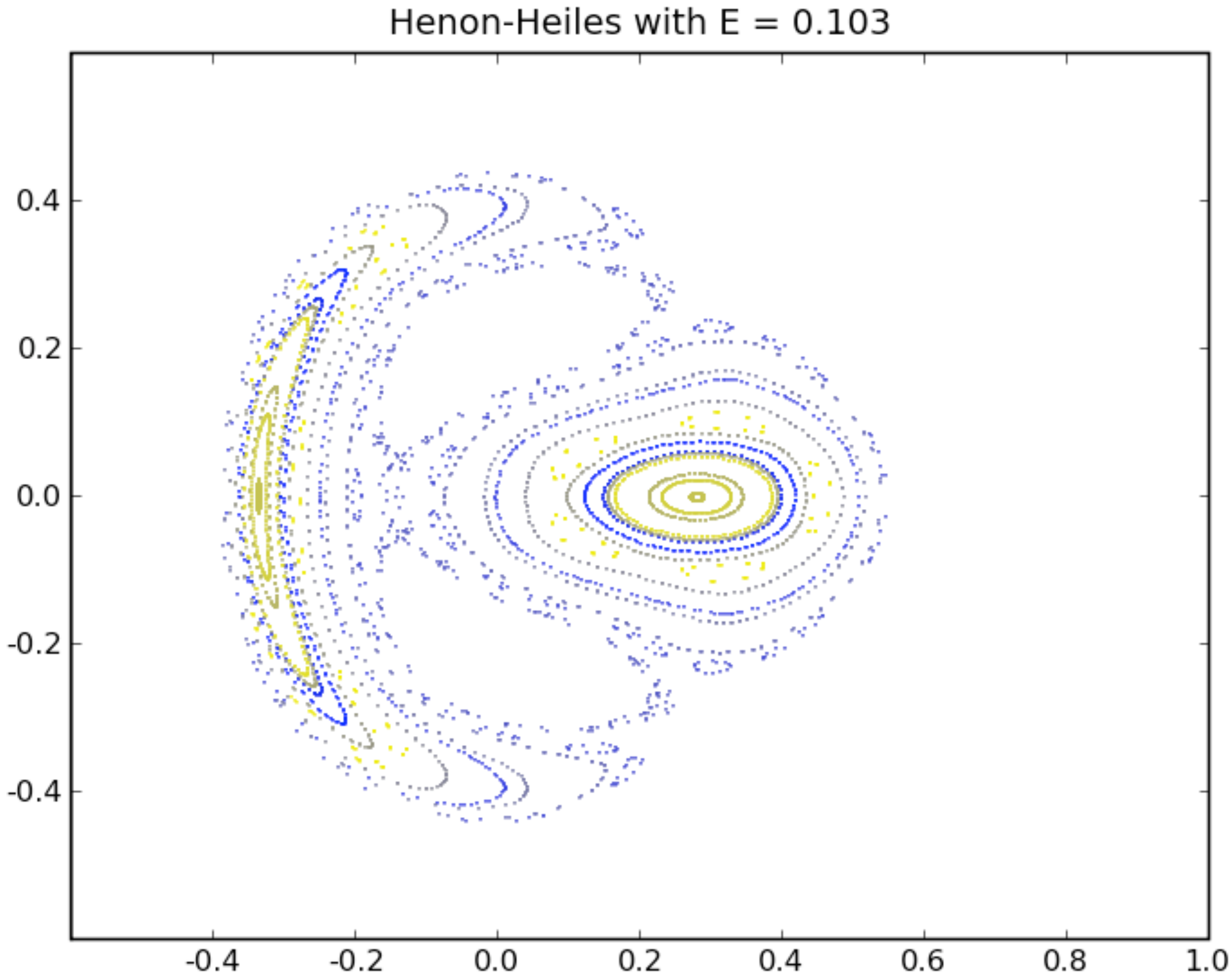
The extremum points I found were

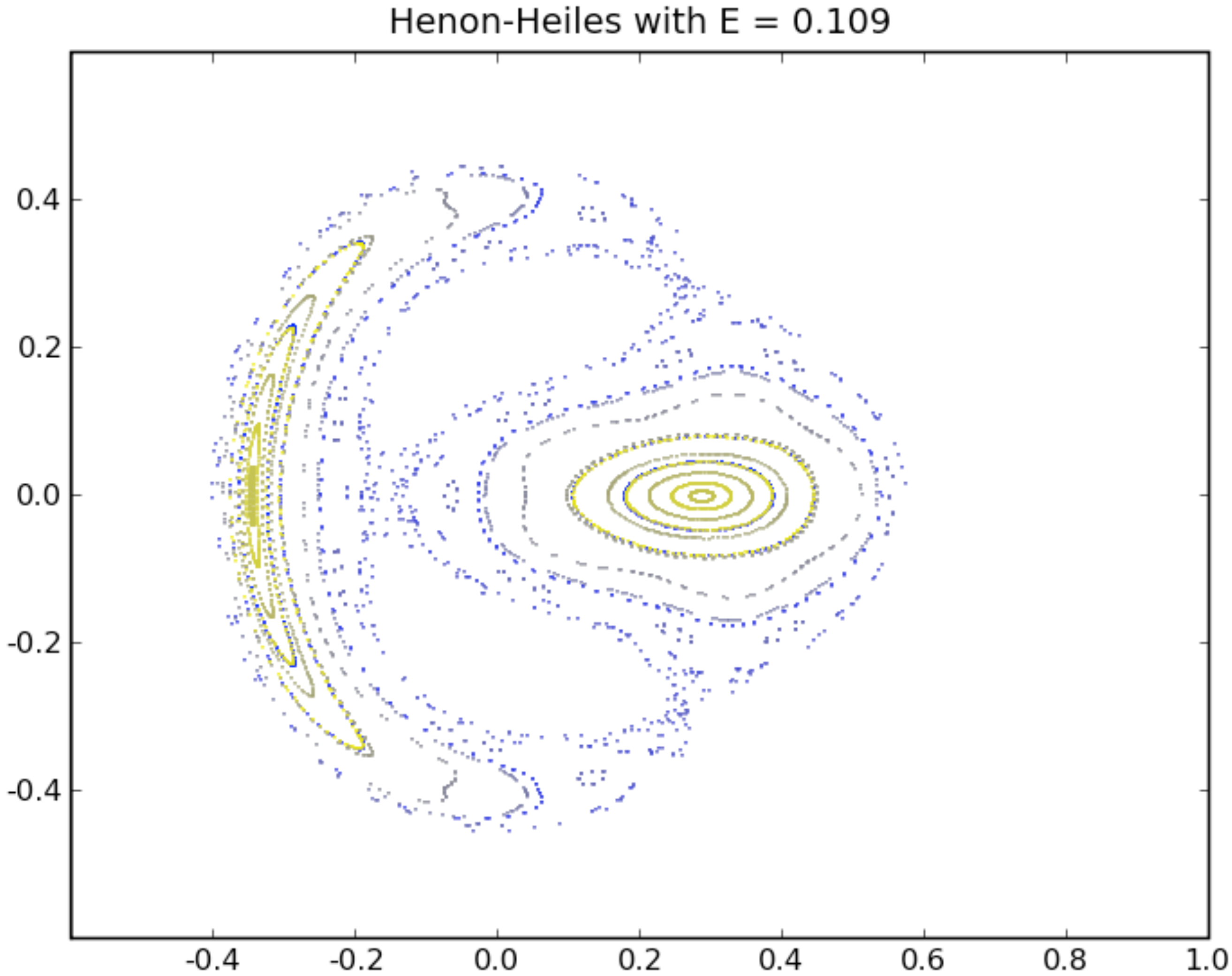
$(0, 0), (0, 1), (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2)$

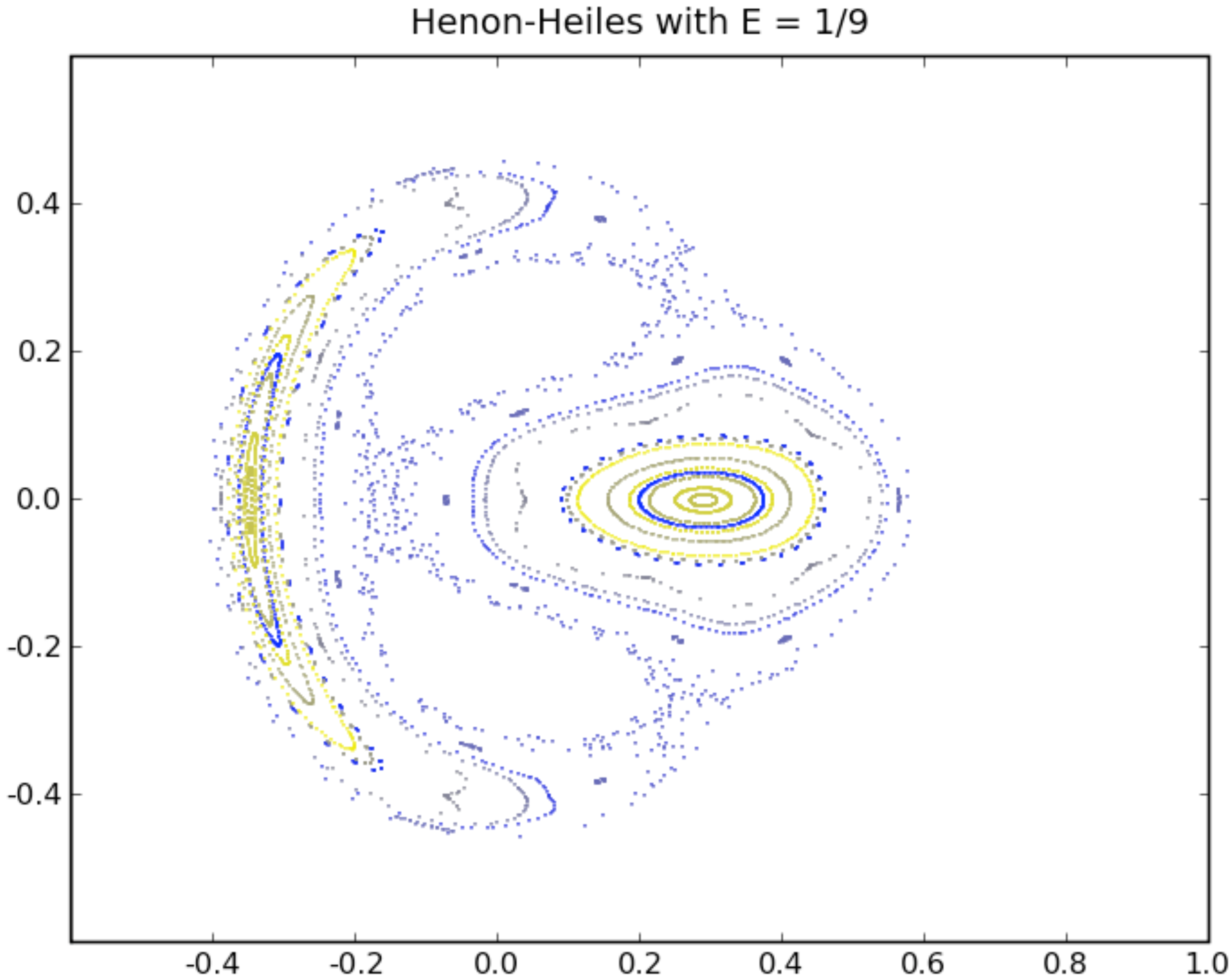
Looking at the dynamics of the system using Poincaré, instead of LCE.

- Show The Mathematica Potential plots
- 3D Python
- Poincaré – Python

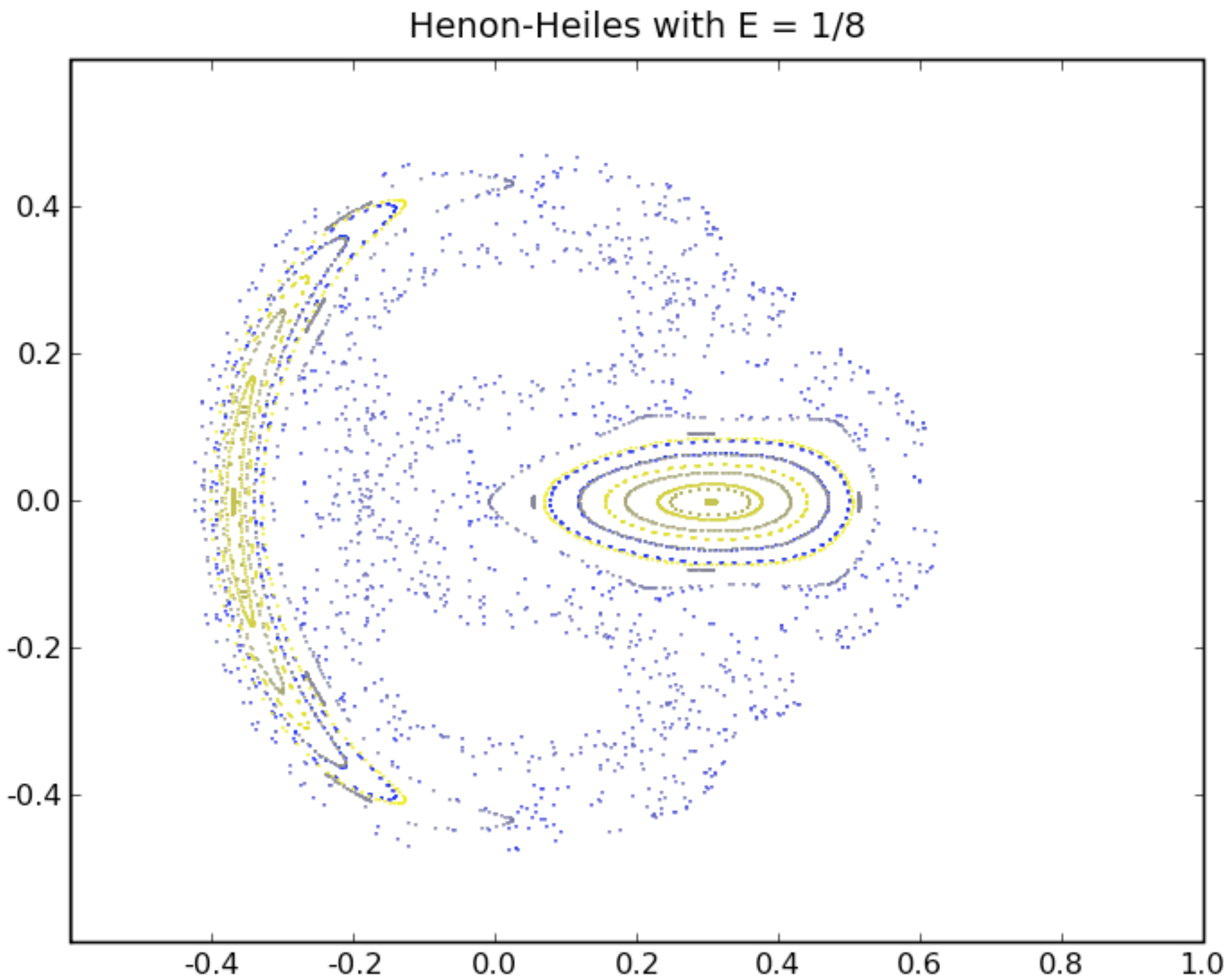

Henon-Heiles with E = 1/12
0.4
0.2
0.0
-0.2
-0.4
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0


Henon-Heiles with E = 0.103
0.4
0.2
0.0
-0.2
-0.4
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0

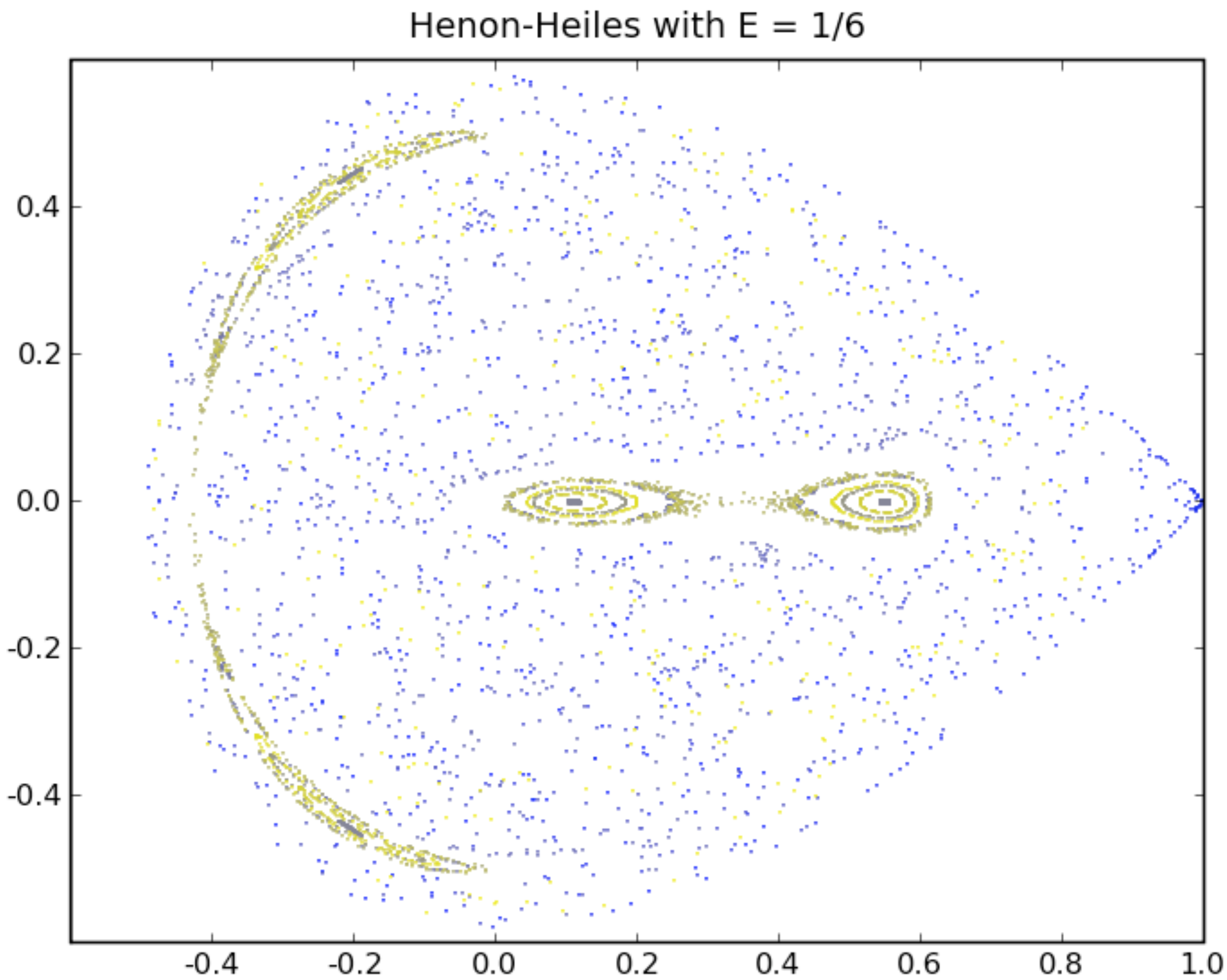

Henon-Heiles with E = 0.109
0.4
0.2
0.0
-0.2
-0.4
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0



Henon-Heiles with E = 1/9
0.4
0.2
0.0
-0.2
-0.4
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0



in



Henon-Heiles with E = 1/6
0.4
0.2
0.0
-0.2
-0.4
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0

More questions

- Going back to the original star motion problem, it would be interesting to know if stars are found in some typical energy or region of the phase space, and to make some Y vs. X plots
- Lyapunov Exponents analysis
- Exploring more initial conditions
- Use a different plane for the Poincaré plot