The Phase-Modulated Logistic Map: an Exploration

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The logistic map is one of the most common examples of how chaotic dynamics can arise in simple systems. In this paper I expand upon the work found in Nandi et al by creating and analyzing bifurcation diagrams at many points in the parameter space and generating a Lyapunov fractal for the space of r^0 *and* r^1*. Many important features of the standard logistic map are preserved in this new dimension and their behavior is analyzed. The video produced from the bifurcation diagram series is used to visually explore the behavior of the attractor as the strength of the feedback phase modulation is varied. The fractal shows at a glance how these structures and features move in parameter space.*

Introduction

It is often desirable to be able to guide a chaotic system into a regime where order is prevalent so that unique behavior can be utilized. The two most common methods at steering a system into a particular behavioral realm are feedback loops and parameter tweaking. In their paper, Nandi et al explore the behavior of the logistic map – among the canonical examples of chaotic dynamics in discrete systems – when its control parameter was varied with a feedback loop.

I wished to explore this system for a number of reasons. First and foremost, the logistic map was the system which enamored me most greatly during the course of the quarter. In this regard, I greatly looked forward to working with it more for my project. Secondly I've had a great appreciation for fractals from an early age thanks to my grandfather's interest in them, notably the Mandelbrot set. In seeing Nandi et al's Lyapunov fractal for their parameter space, I was inspired to create one of higher resolution, and plotting the actual Lyapunov exponent rather than simply the sign of the exponent. More concretely, Nandi et al discuss the evolution of some features into the r^1 dimension but I found much of this unsatisfactory since they merely plotted trajectories in parameter space which gives me no idea of the actual dynamics of these features and structures in the bifurcation diagram.

To achieve these goals I implemented the map and used it and some simple analysis code. The vast majority of my results fall into two pieces of media: a movie 1001 bifurcation diagrams – one each for r^1 from 0 to 1 in steps of 0.001; and secondly a 1024x1024 pixel image of the Lyapunov fractal showing slightly more detail than Nandi et al as well as a version in a color scale to show the value of the Lyapunov exponent.

Background

The standard logistic map is a one dimensional discrete map which obeys the following iteration equation:

$$x_{n+1} = r\,x_n(1-x_n)$$

where $x_n \in [0,1]$ and $r \in [0,4]$. This simple mapping already exhibits fantastic dynamics, quickly summarized here. The dynamics of the evolution of the map can be quickly seen in a cobweb plot (figure 1). The range of dynamical behavior with respect to the value of r is seen in the bifurcation diagram (figure 2), which is read as being a series of histograms as seen from above and arranged vertically, so that each vertical slice is the histogram corresponding to the visitation probability of the pixel by the map for that particular value of r. There are many important features of the bifurcation diagram which are important to point out at this time: the value of r at which the period doubling cascade culminates in a chaotic regime: 3.57; secondly the value of r at which the band merging occurs: 3.68; and lastly the value of r at which a period-3 superstable orbit exists: 3.83. Lyapunov exponents (figure 3) numerically describe the degree of chaos in the map at a particular value of r: values less than zero correspond to ordered behavior, thus values of r where the bifurcation diagram shows simple periodicity have such Lyapunov values; values greater than zero are chaotic regimes where aperiodic orbits prevail and the behavior is visually complex.

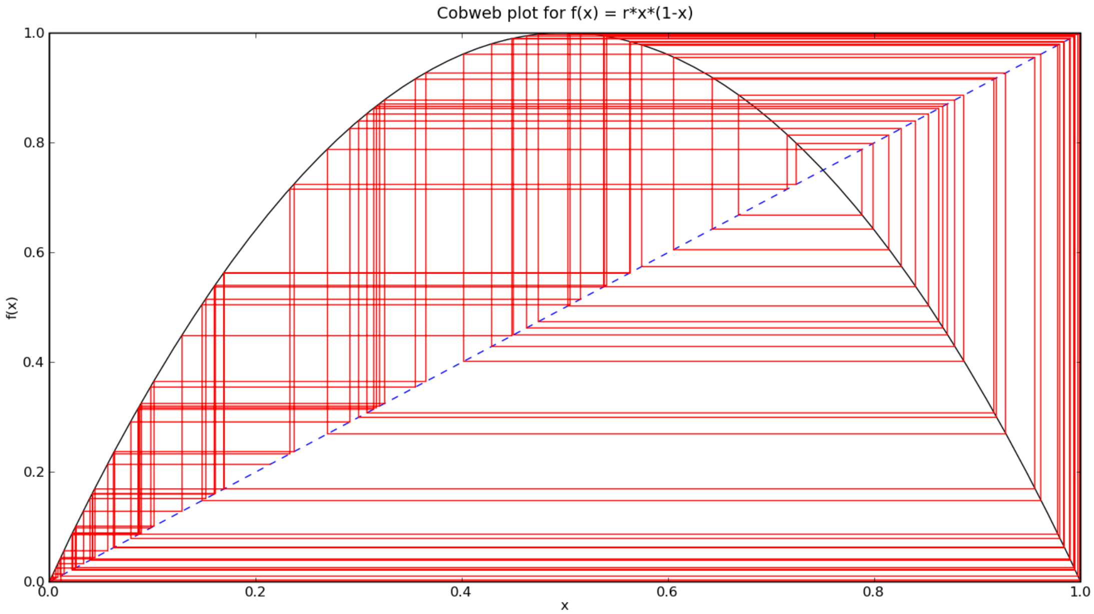


Figure 1: Cobweb plot showing the complex iterative behavior of the logistic map in a chaotic regime.

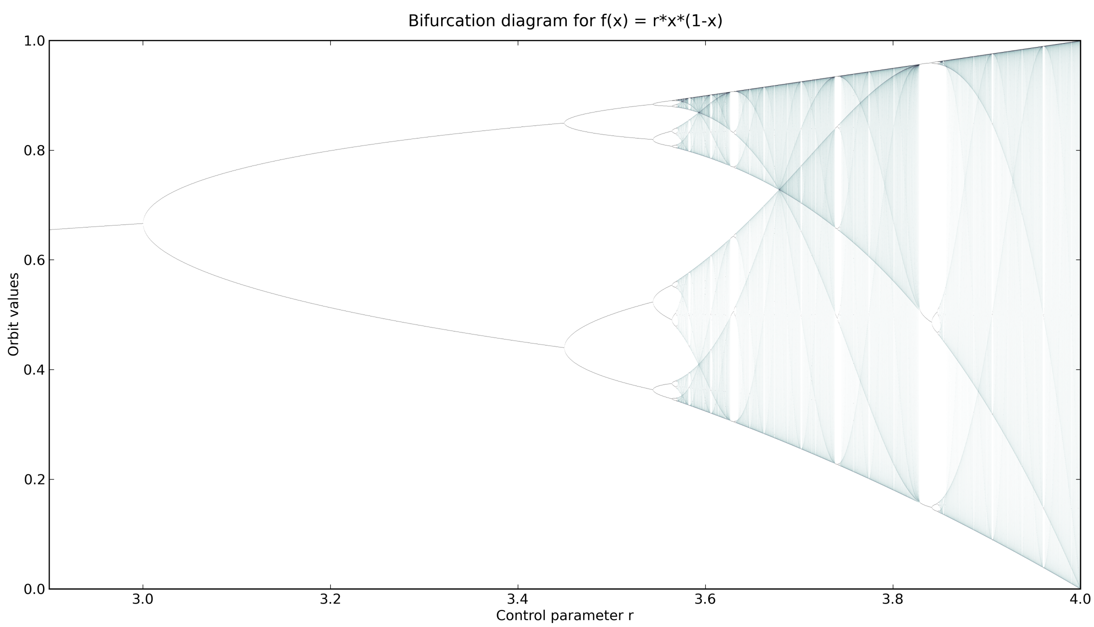


Figure 2: Bifurcation diagram showing the range of behavior for the logistic map at various interesting values of r. Of note in this diagram are the period-3 superstable orbit at $r \sim 3.83$, the band merging at $r \sim 3.68$, and the onset of chaos at the end of the period-doubling cascade at $r \sim 3.57$.

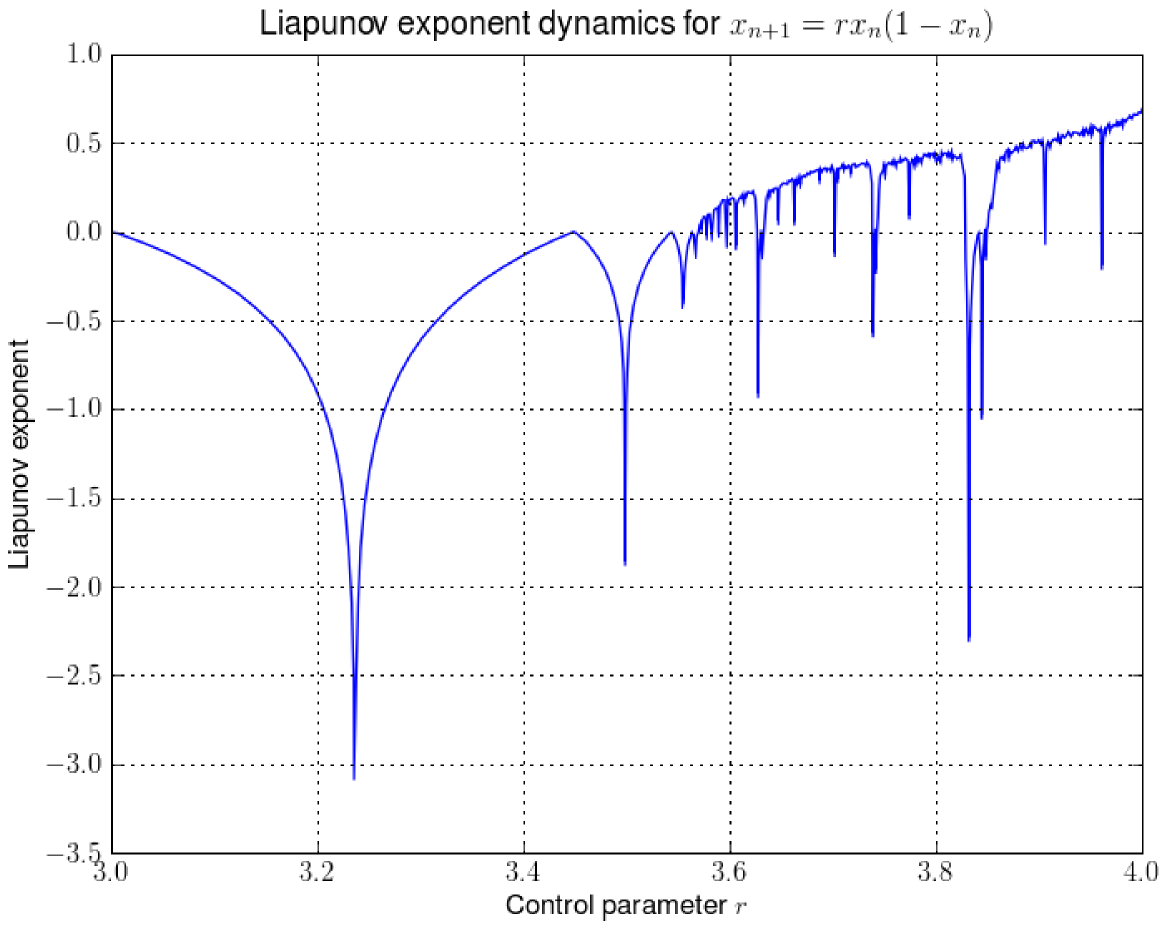


Figure 3: Features from the bifurcation diagram can also be seen here, most notably the period 3 window.

Dynamical System

The phase-modulated logistic map follows a similar iteration equation, but due to the dependance on both x_n and x_{n-1} the phase-modulated logistic map requires two dimensions:

$$x^0_{n+1} = r\, x^0_n (1 - x^0_n)$$

$$where\; r = r^0 + (4 - r^0)\, r^1\, sign(x^1_n - x^0_n)$$

$$x^1_{n+1} = x^0_n$$

Clearly x^1 holds the previous value of x^0, fulfilling our need of immediate history. The only other difference is that r has become a function of two parameters, r^0 and r^1 and of the direction of motion of x, while the entire second term has been scaled to prevent r from exceeding 4. The r^0 term plays the same role here as did r in the standard logistic map, and r^1 controls how strong the phase modulation is. Here, $r^0 \in [2,4]$ and $r^1 \in [0,1]$ while again $x^0_n, x^1_n \in [0,1]$. Since r depends solely on the direction of motion of x, it results in only two different iteration functions for x^0_{n+1}: $r = r^0 + r^1$ if $x^0_n < x^0_{n-1}$ and $r^0 - r^1$ otherwise. This will be manifest in the bifurcation diagrams.

Methods

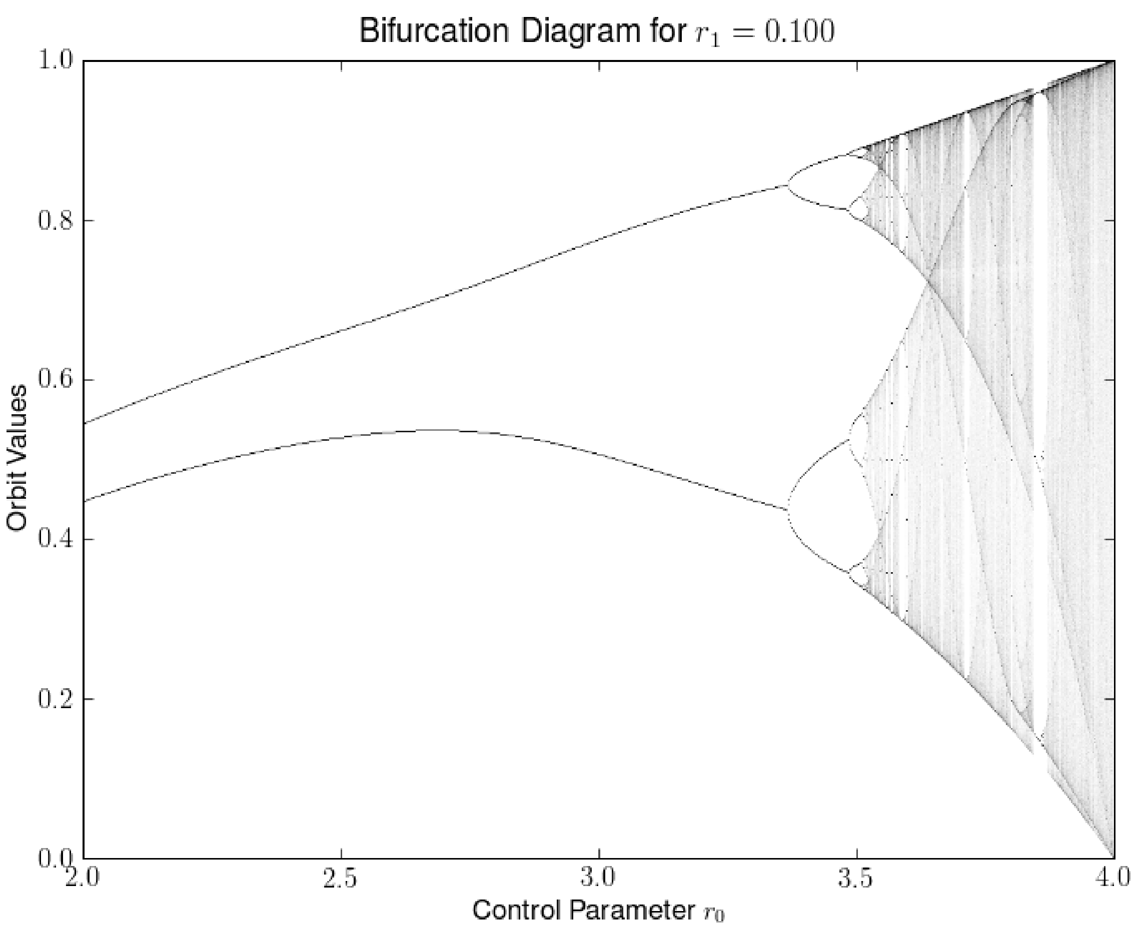
To analyze the system, fairly succinct Python code was written. The majority of the heavy lifting here is due to numpy. The primary visualization methods are bifurcation diagrams and Lyapunov fractals.

The bifurcation diagrams are created by instantiating a map for each value of r^0 required and taking a long trajectory. A histogram of this trajectory is then computed at the bin resolution requested. These histograms are then stacked and displayed as an image using matplotlib. Each bifurcation diagram at a resolution of 1024x768 takes ~5 minutes, meaning a 1001 frame film takes ~3.5 days of computation. The open source video encoder mencoder was used to stitch these frames into a movie file, compressed somewhat for space concerns.

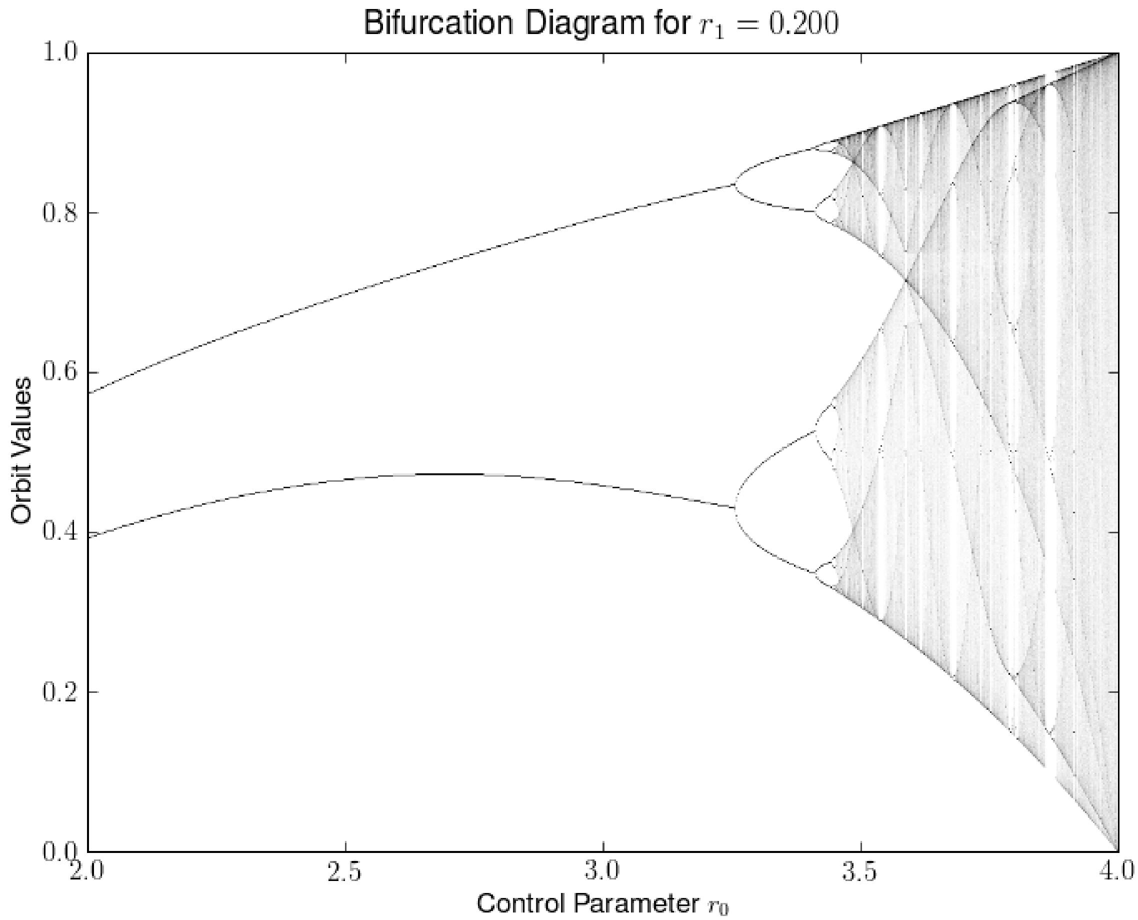
The Lyapunov fractal is created by computing the Lyapunov exponent for each pixel in the image and then displaying. Lyapunov exponent calculation is done numerically using the standard formulation. Using pure Python, the fastest I could get a general Lyapunov exponent to be calculated was ~0.0174 seconds, which results in ~5 hours for a 1024x1024 image.

Results

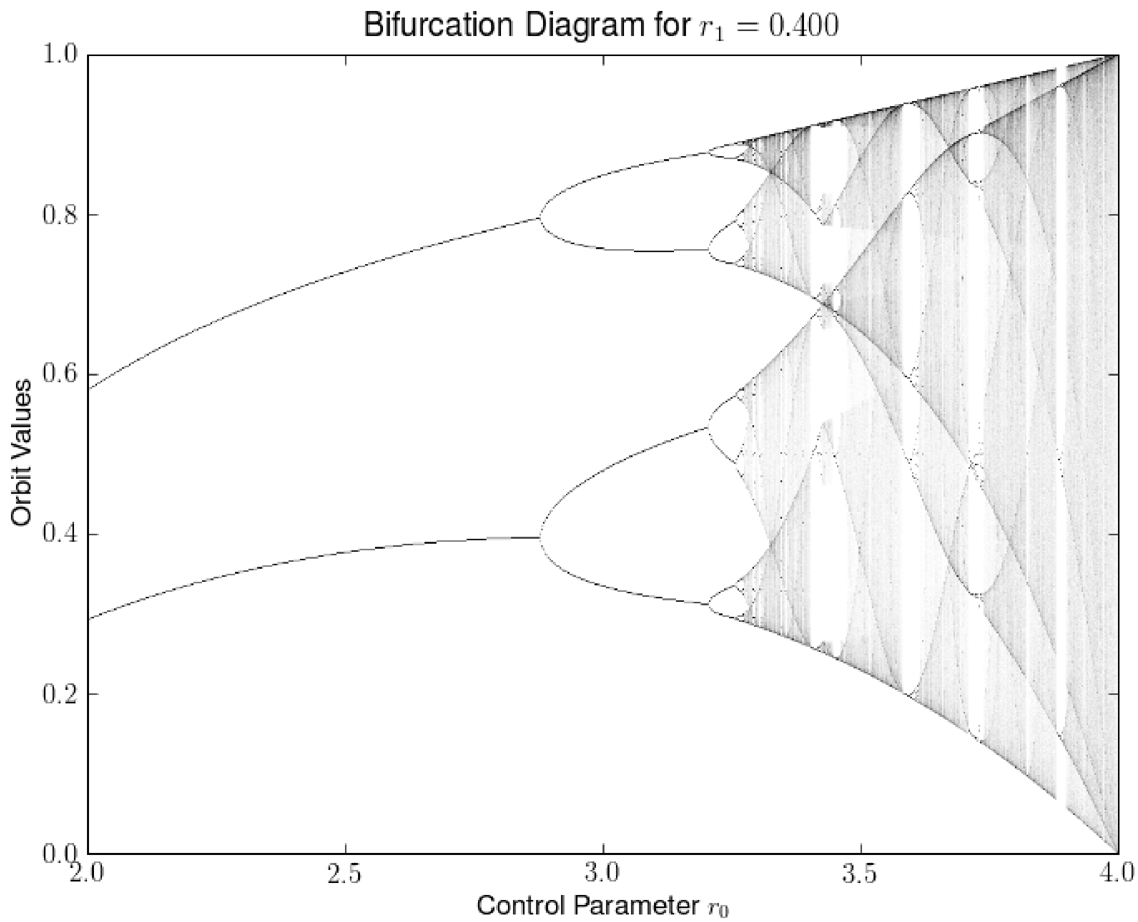
In the following series of bifurcation diagrams, I will note the dynamics of many features.



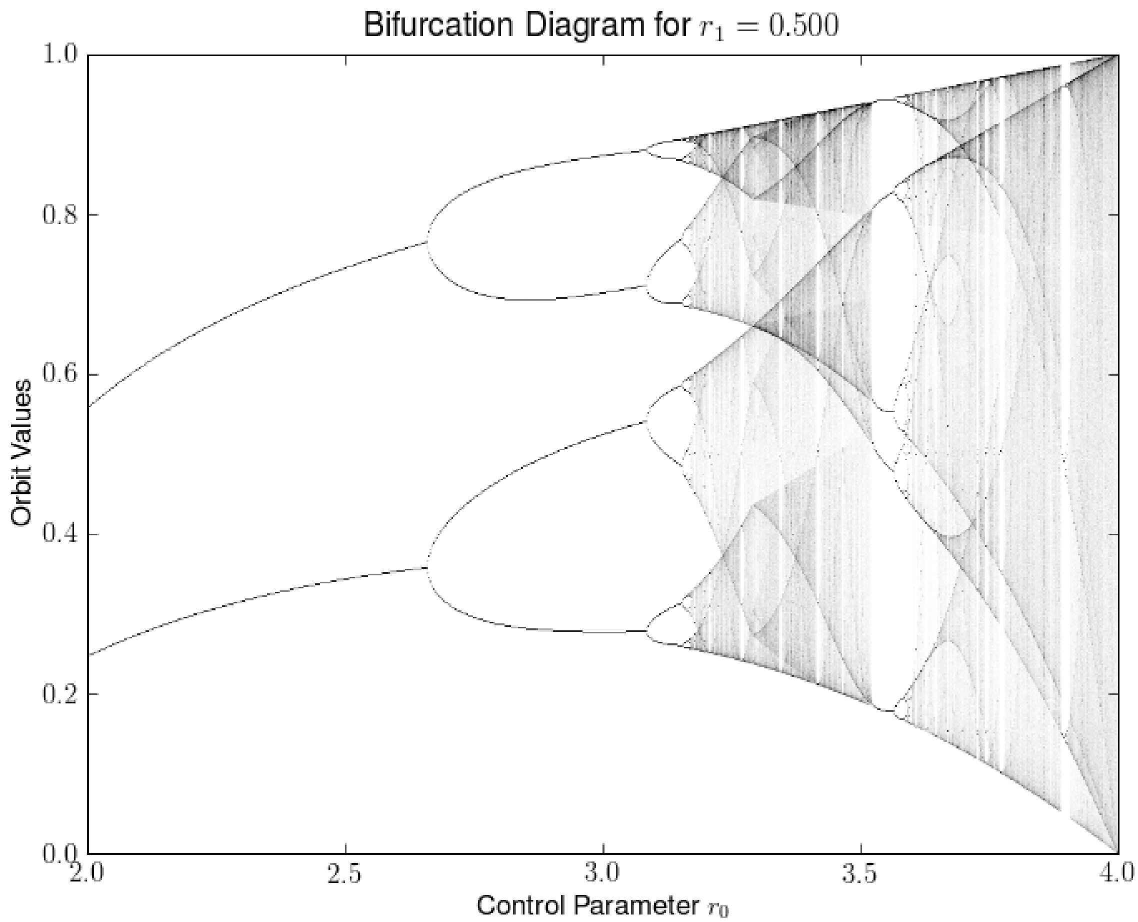
The most striking feature is that the first period doubling has shifted far to the left after a relatively small variation in r^1. Also of note is that the apparent splitting of veils is occurring – double sets most easily seen at the top and bottom of the period 3 orbit.



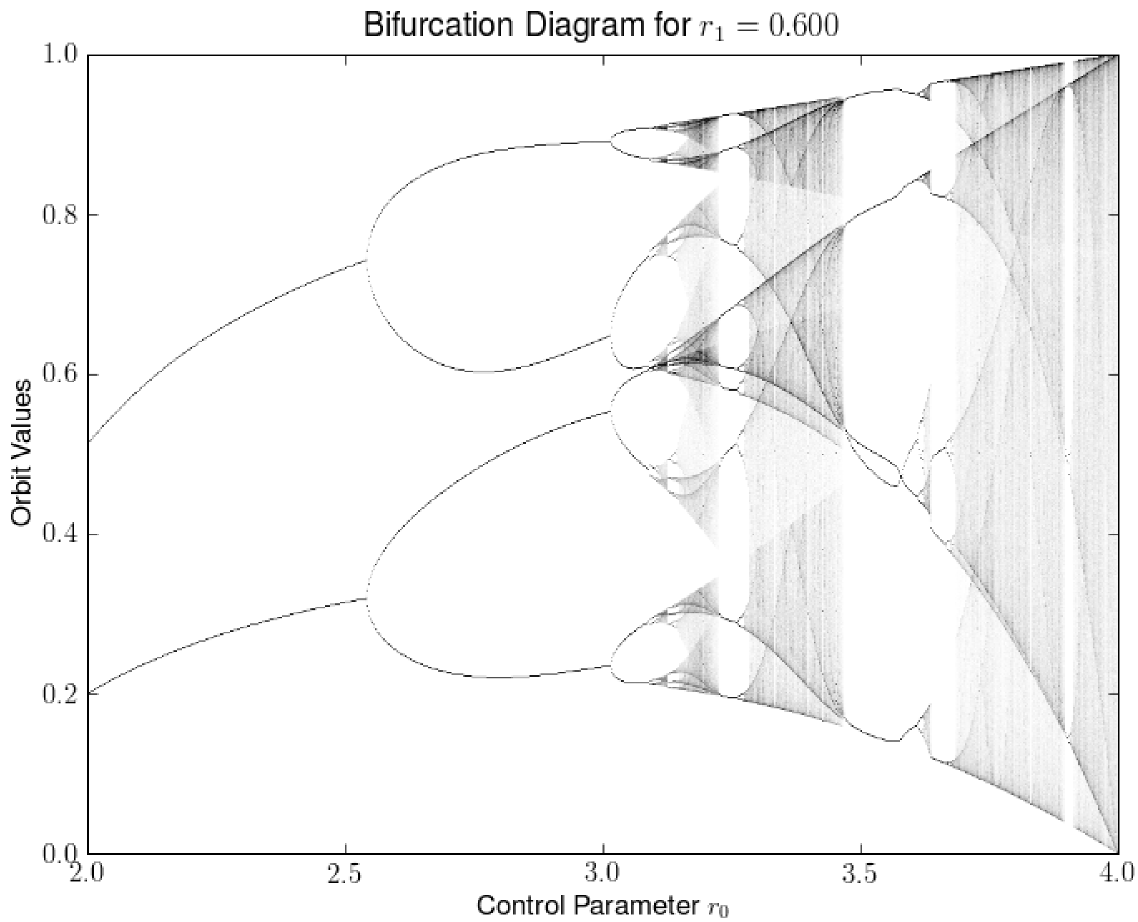
Here at $r^1 = 0.2$ the veil splitting is even more obvious, and in particular separation of veils at the band merge. It appears as though some veils 'break off' from the band merging, a point where all veils intersect in the standard logistic map. The likely cause of this veil splitting is that each veil is the result of one of the two iteration functions: $r^0 + r^1$ or $r^0 - r^1$.



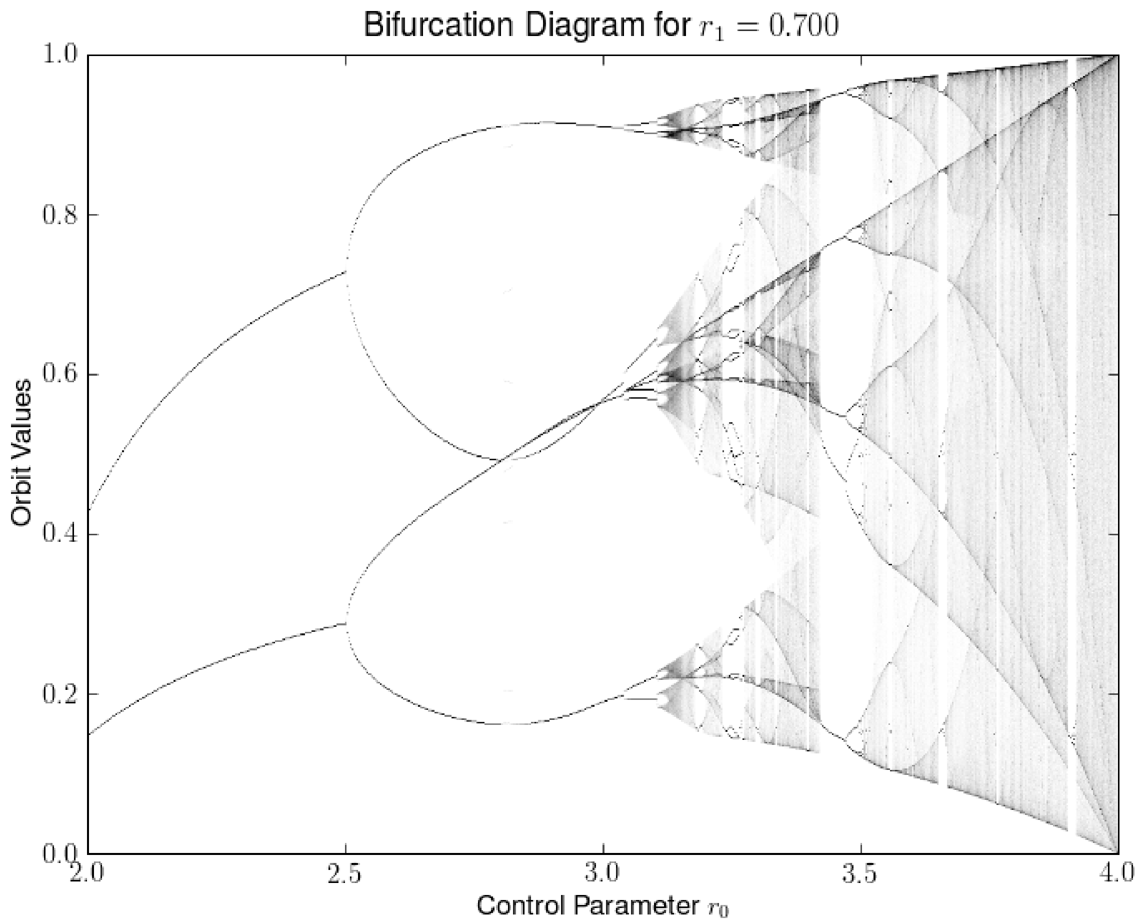
By this point, many new periodic windows have begun appearing. The increased numbers are likely due to veils from both iteration functions working together to cause the windows of order.



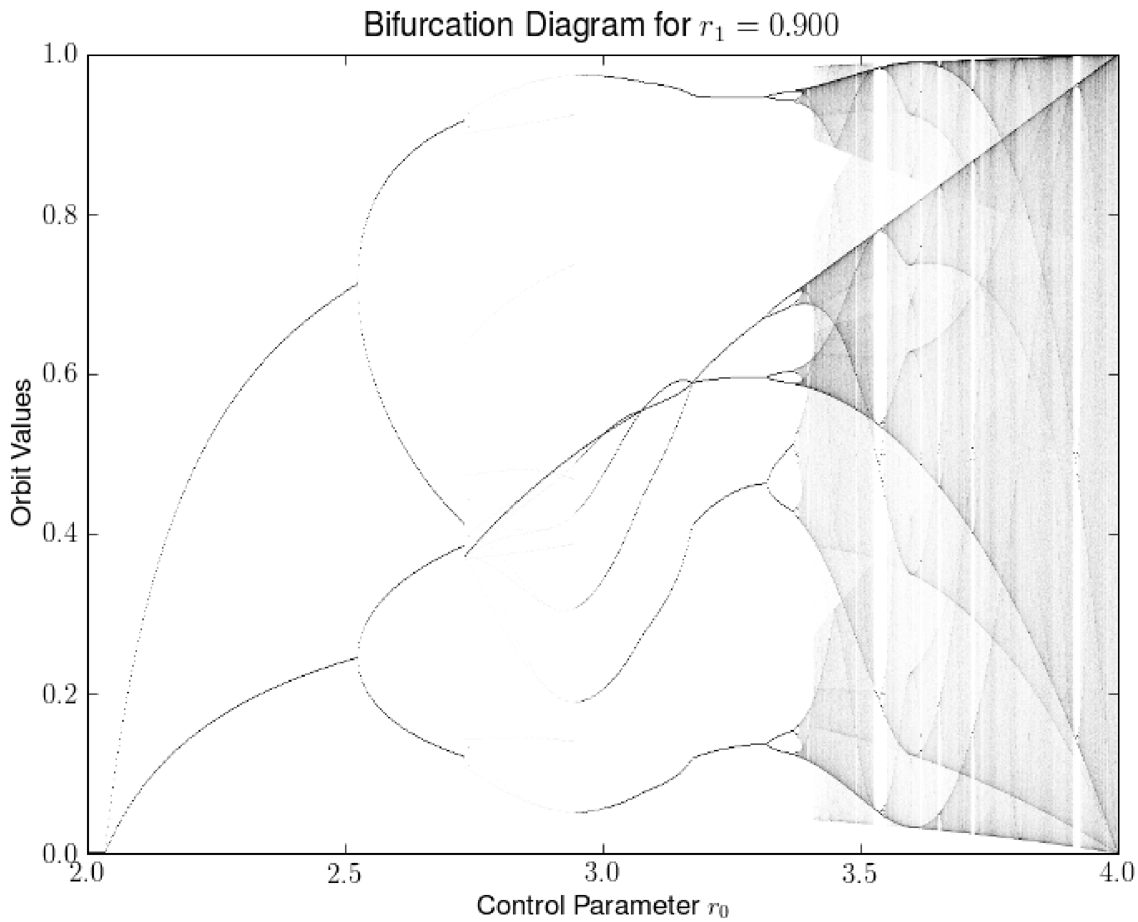
At r^1 = 0.5, the period 5 window has become a very large feature, dominating a large fraction of the chaotic regime. The period 3 window has been relegated to a smaller width further to the right. Also by this point the period 4 bifurcations have become distorted from their parabolic shape of yore and the upper one has become distinctly horseshoe shaped. Lastly, 'shadows' of density can be seen behind the separating veils from the band merging.



The bifurcations have become increasingly distorted, and the period 8 ones appear to, in the middle, have collided with the band merging. The period 5 window continues to grow.

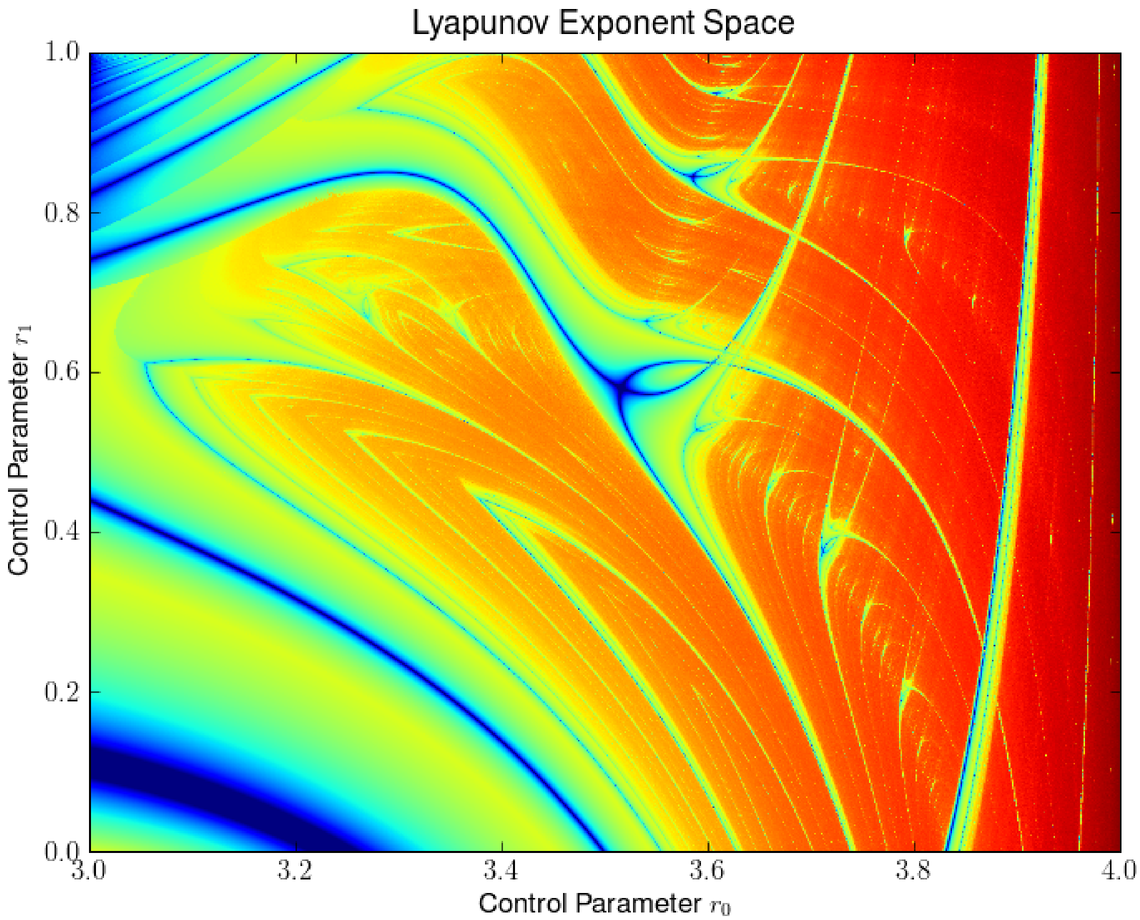


Finally at $r^1 = 0.7$ the period 4 bifurcations have overlaps, resulting in a period incrementation. Beyond this, the entire chaotic regime between what was previously the period doubling cascade and the period 5 window has become home to wandering periodic windows.



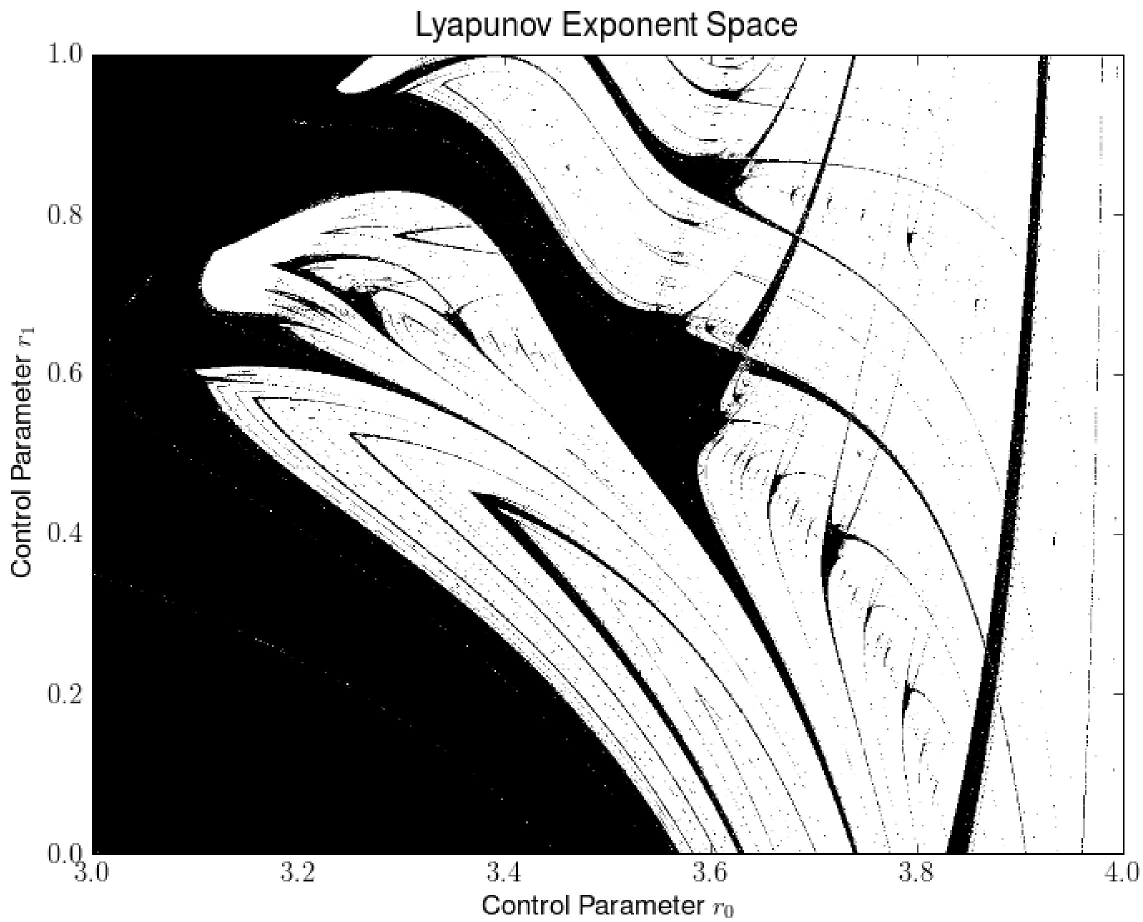
At this point the period 5 window has merged with the ordered bifurcation regime on the left, eliminating the chaotic area between them.

All these features and their dynamics can more easily be seen in the Lyapunov fractal.



Many major features and their dynamics can be seen in this image. The blues and greens correspond to a negative Lyapunov exponent, while the oranges and reds correspond to positive values. The blue and green streak going up the right side of the image and narrowing as it moves slightly right is the period 3 window. The tips pointed arches that begin on the r^0 axis between ~3.6 and ~3.7 lay on the band merge. It would seem that periodic windows have a tendency to merge at the band merge. Further window merging occurs in the ~3.8 region with r^1 between 0.1 and 0.5. The large blue & green arm which begins at r^0 ~ 3.75 and grows, eventually having some complicated dynamics before merging with the general ordered regime on the left is the period 5 window.

In the spirit of the original paper, this image instead rendering the sign of the Lyapunov exponent reveals which areas of parameter space have ordered and chaotic behavior.



This image shows the fairly large island of order along the period 5 window which may be of use in an application.

Conclusion

Upon viewing this data, in particular the bifurcation movie, one can build a more intuitive feel for how this map behaves as r^1 is varied. Highlighted are the dynamics of a few notable structures in the standard logistic map and how they have highly unintuitive or complex behavior in this phase-modulated logistic map. The most unexpected is the prominent role the period 5 windows takes, striking a wedge in the chaotic regime and eventually eliminating the leftmost half when it merges with the period doubling side.

Bibliography

The Phase-Modulated Logistic Map - Amitabha Nandi, Debabrata Dutta, Jayanta K. Bhattacharjee, and Ramakrishna Ramaswamy, Chaos 15, 023107 (2005), DOI:10.1063/1.1914755