The Big, Big Picture (Bifurcations II)

Reading for this lecture:

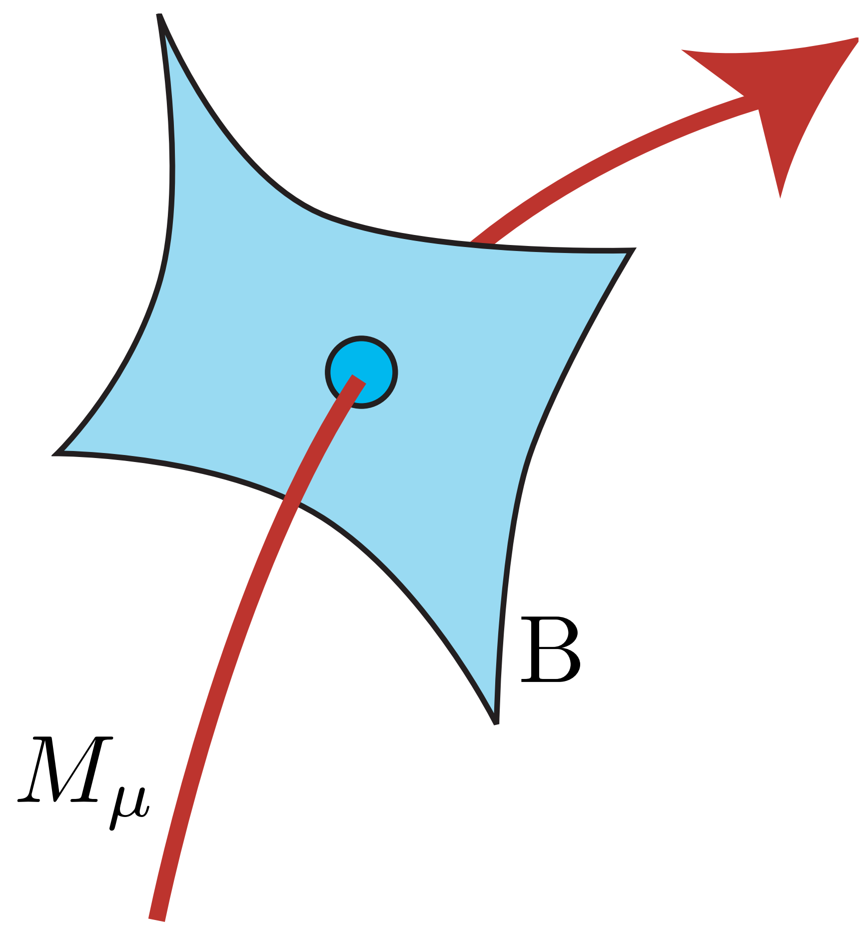
NDAC, Chapter 8 and Sec. 10.0-10.4.

The Big, Big Picture (Bifurcations II) ...

Beyond fixed points:

Bifurcation: Qualitative change in behavior as a control parameter is (slowly) varied.

Today: Bifurcations between time-dependent behaviors



The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps:

Logistic map: $x_{n+1} = r x_n (1 - x_n)$

State space: $x_n \in [0, 1]$

Parameter (height): $r \in [0, 4]$

Maximum: $x = \frac{1}{2}$

Fixed points x^* such that:

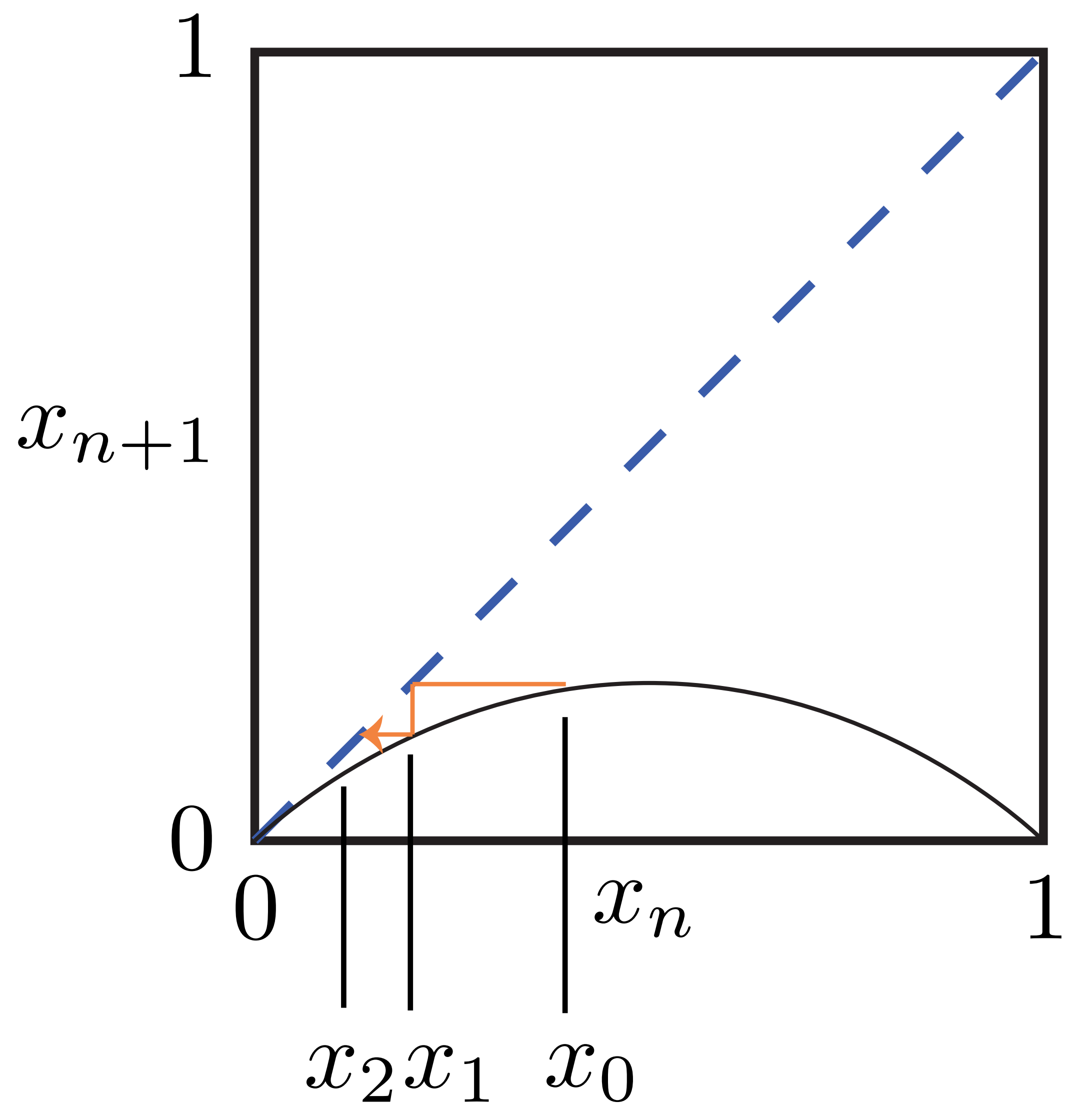
$$x^* = f(x^*)$$

Logistic map:

$$x^* = 0, \ \forall r$$

Stable for $0 \leq r < 1$:

$$|f'(x^*)| < 1$$



The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

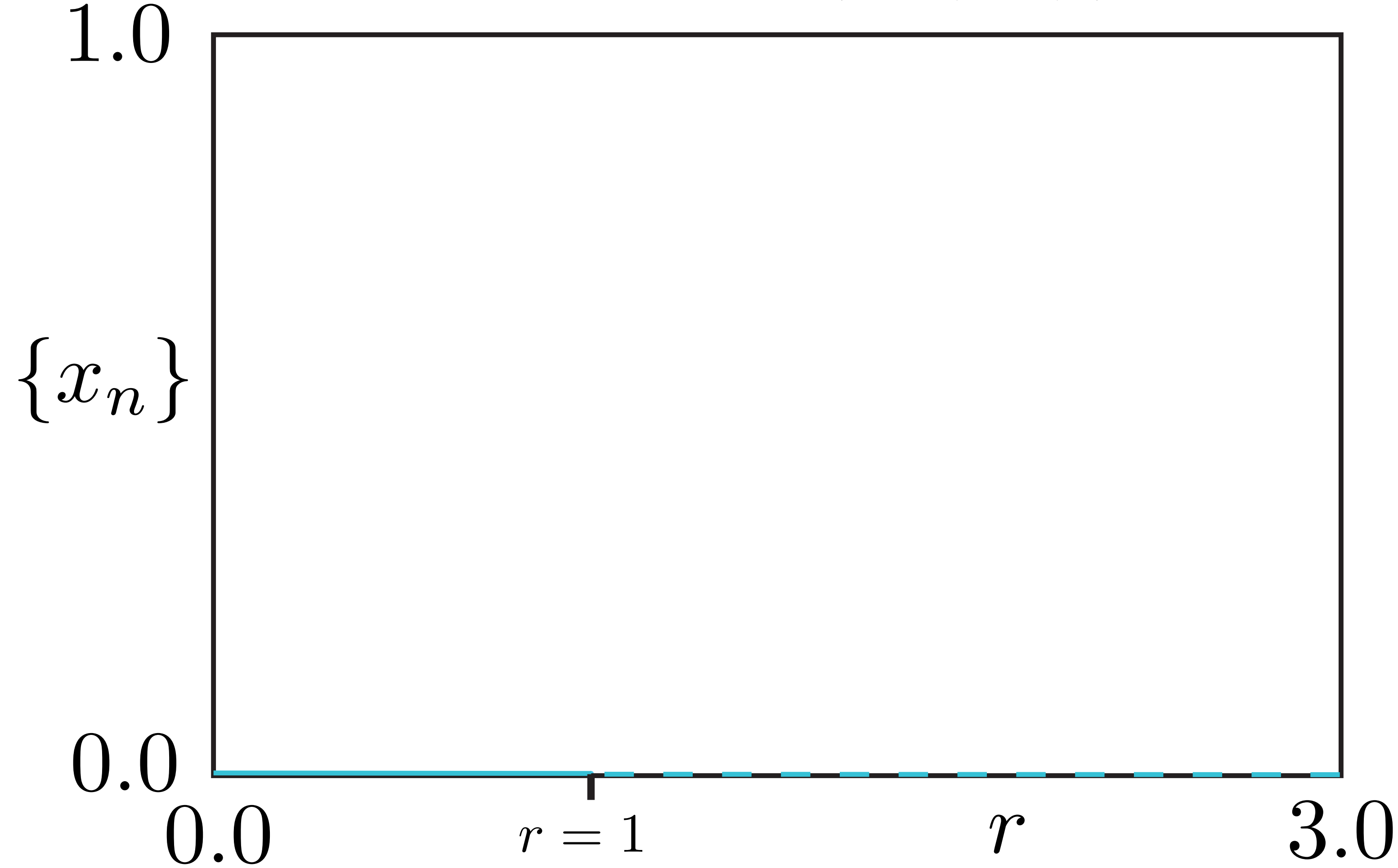
But the fixed point at $x^* = 0$ goes unstable:

$$f'(x) = r - 2rx$$

$$x^* = 0 \Rightarrow f'(x^*) = r$$

Bifurcation diagram view:

$|f'(x^*)| < 1, \ 0 \leq r < 1$ $\qquad$ $|f'(x^*)| > 1, \ r > 1$



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The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

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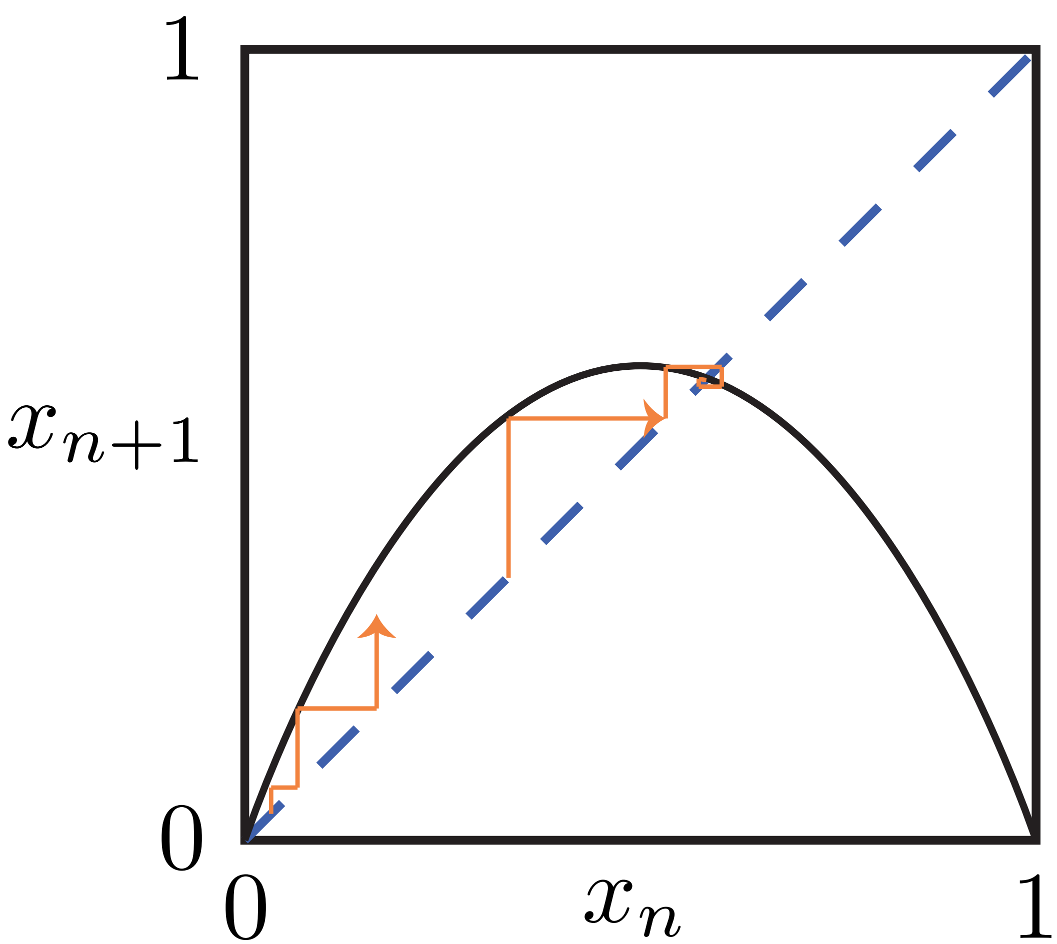
However, for $r > 1$ there is another fixed point: $x^* = 1 - r^{-1}$

At what parameter value?

Where the other loses stability



$$|f'(x^*)| = 1$$

$$f'(x) = r - 2rx$$

$$x^* = 0 \;\Rightarrow\; f'(x^*) = r$$

x^* is unstable when $r \geq 1$

The Big, Big Picture (Bifurcations II) ...

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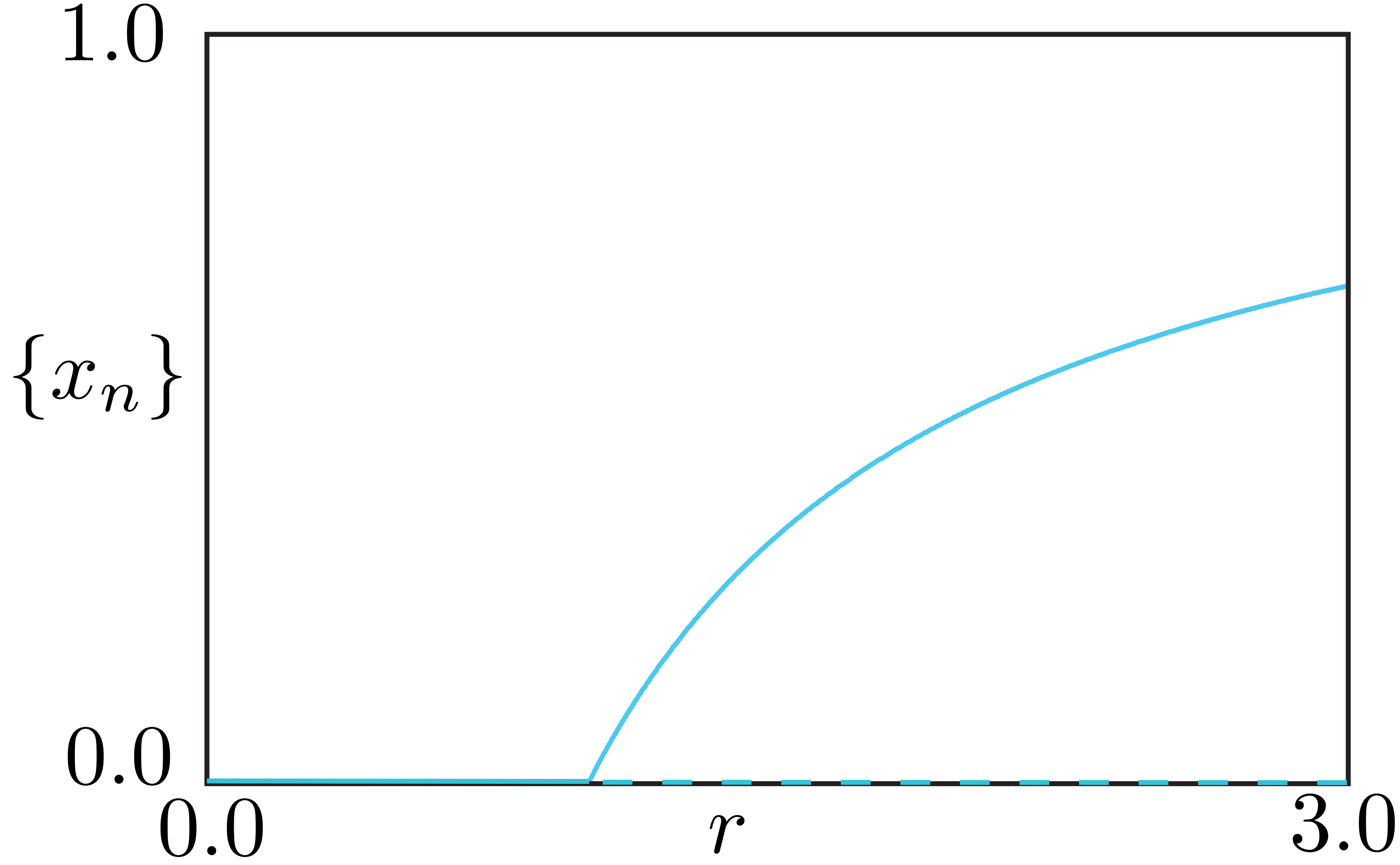
Logistic map ...

The other fixed point: $x^* = 1 - r^{-1}$

Bifurcation diagram view:

$$|f'(x^*)| < 1, \; 1 < r < \; ?$$



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The Big, Big Picture (Bifurcations II) ...

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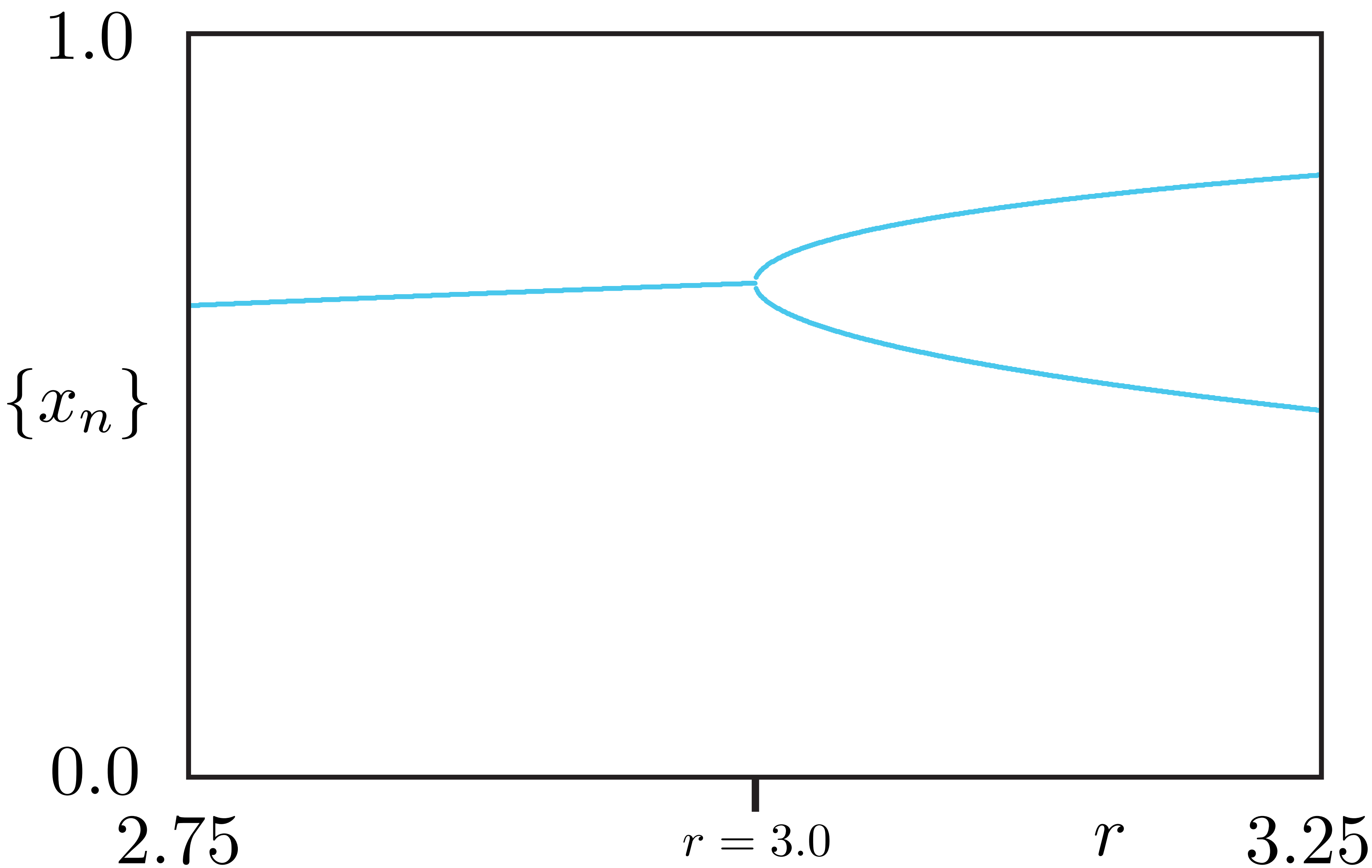
Fixed point (period-1) to period-2 limit cycle



Did period-1 fixed point disappear?

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

At what bifurcation parameter value is P-2 orbit stable?

P-2 orbit: $\{x_1^*, x_2^*\}$

$$x_1^* = f(x_2^*) = f \circ f(x_1^*)$$

Fixed point: $x_1^* = f^2(x_1^*)$

Calculate: $x^* = rf(x^*)(1 - f(x^*))$

$$x^* = r^2 x^* (1 - x^*)(1 - rx^*(1 - x^*))$$

Find parameter such that this quartic equation has solutions!

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Simpler: When does nontrivial P-1 go unstable?

P-1: $x^* = 1 - r^{-1}$

Slope: $f'(x) = r(1 - 2x)$

Slope at fixed point: $f'(x^*) = 2 - r$

Marginally stable: $|f'(x^*)| = 1$

$$|2 - r| = 1$$

Two solutions:

First, P-1 to P-1 bifurcation: $r = 1$

What we're asking about:

P-1 to P-2 bifurcation: $r = 3$

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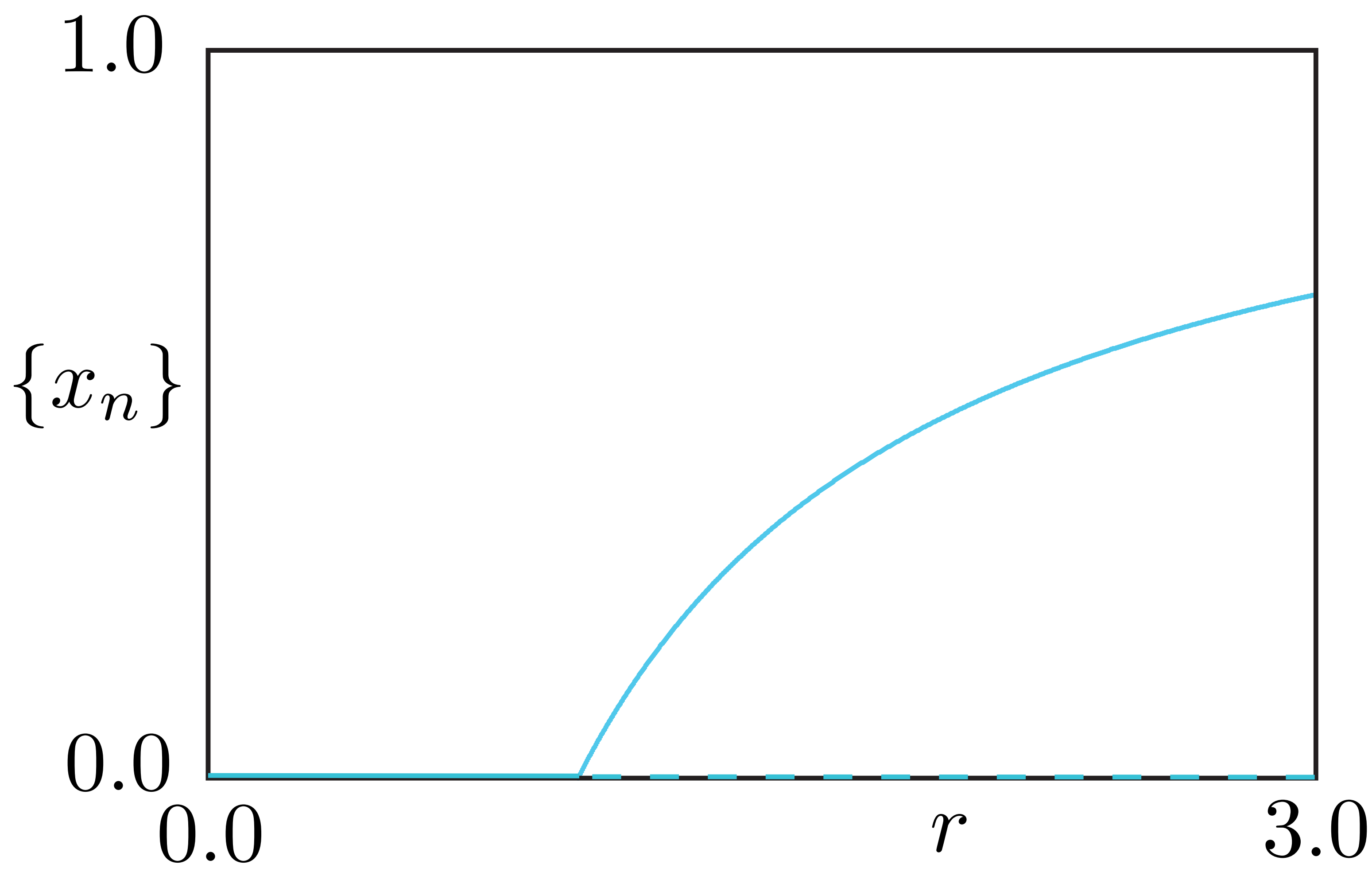
Bifurcation Theory of 1D Maps ...

Logistic map ...

Let's review:



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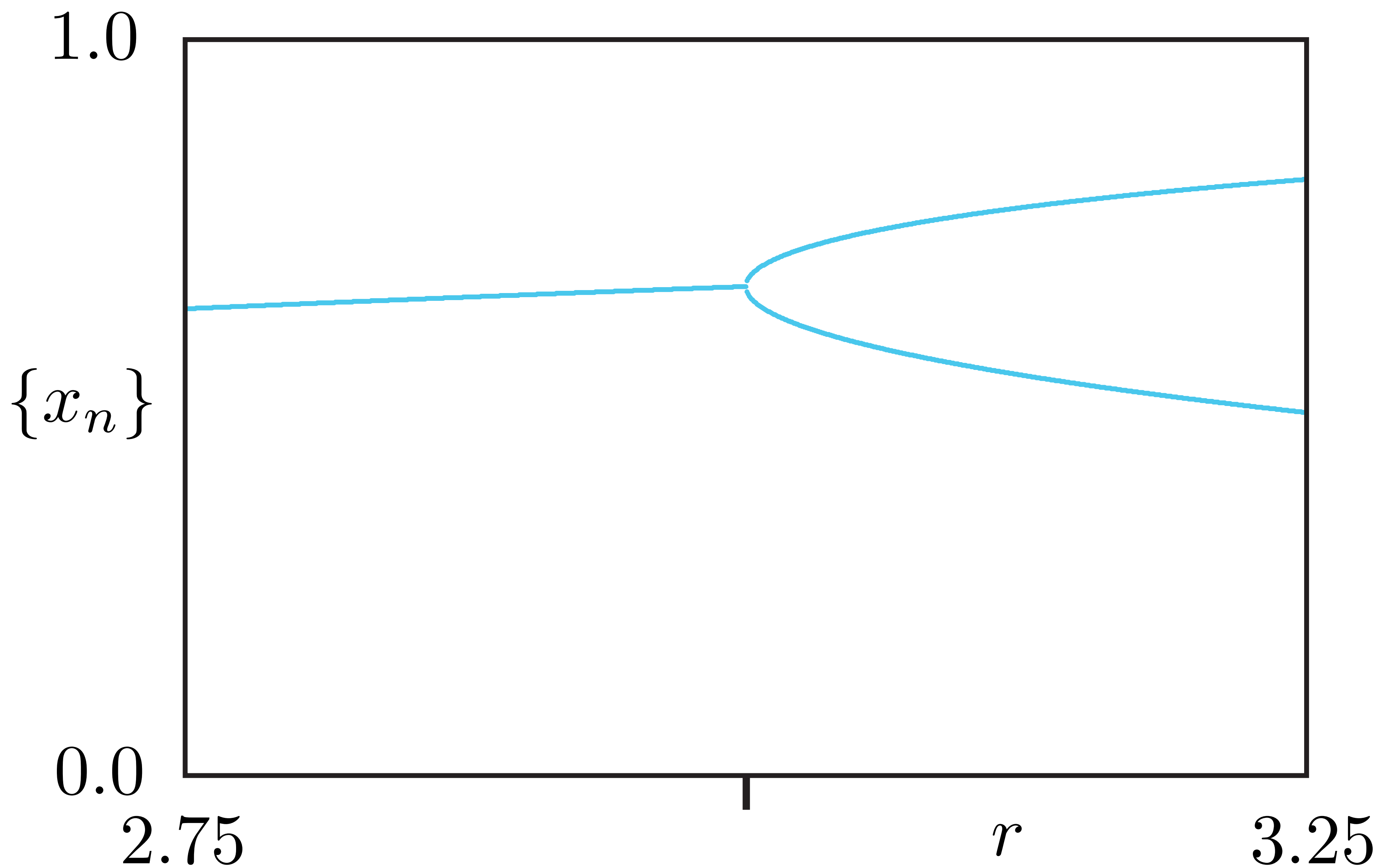
Bifurcation Theory of 1D Maps ...

Logistic map ...

Review ...



Bifurcation Theory of 1D Maps ...

Logistic map ...

1. P-1 to P-1: origin goes unstable
2. P-1 to P-2: nontrivial fixed point goes unstable
3. ...

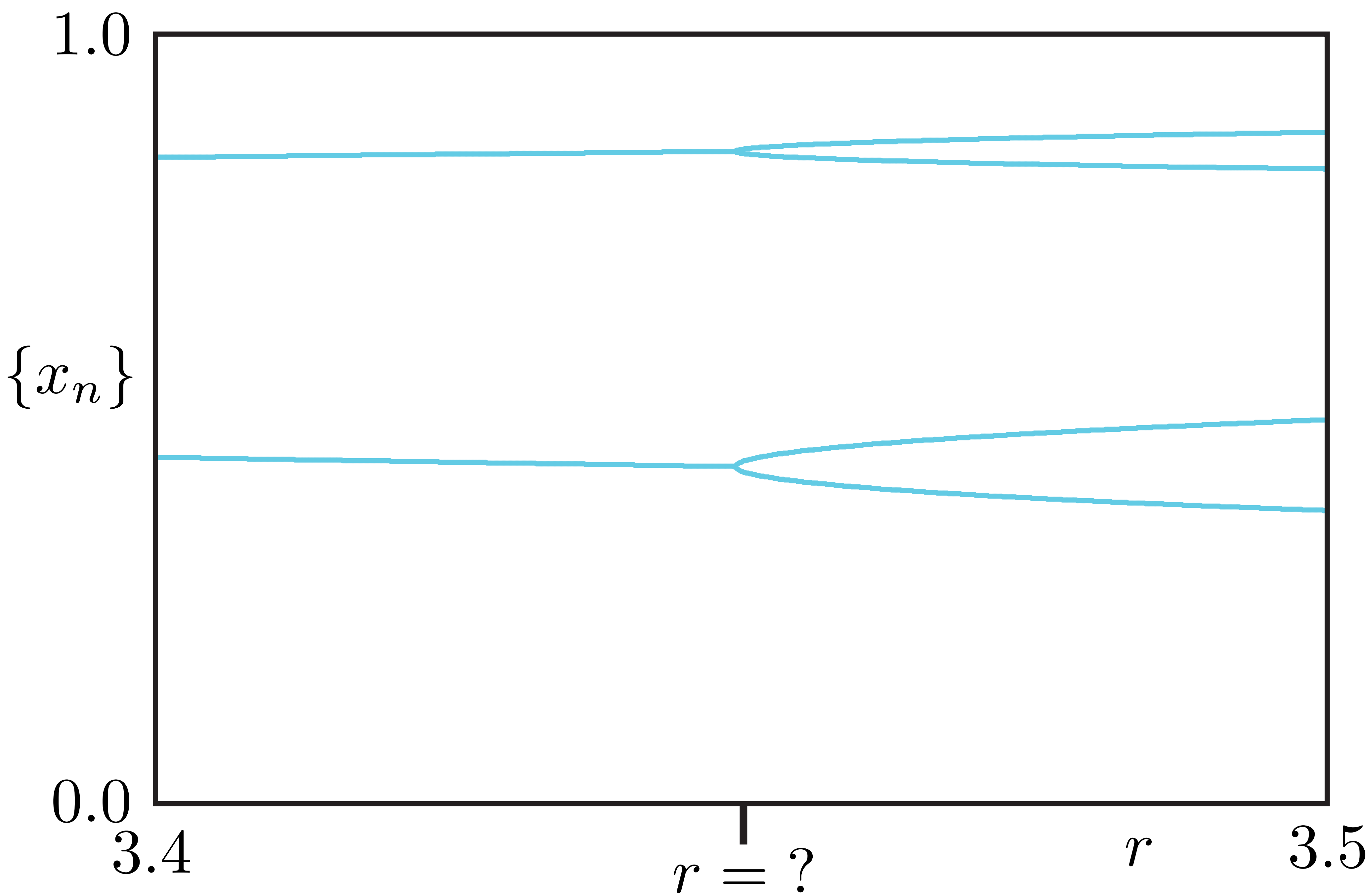
What's next as we increase r ?

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Limit cycle to limit cycle: Period-2 to Period-4



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Bifurcation Theory of 1D Maps ...

Logistic map ...

What parameter value?

Way too messy ... solve numerically:

Period-p limit cycle: $x_1 \to x_2 \to \cdots x_p \to x_1$

Criteria:

Fixed points of p-iterate: $x_i = f^p(x_i), \; i = 1, \ldots, p$

Onset of instability: $\left| \frac{d}{dx} f^p(x) \right| = 1$

Stability along the orbit:

$$\left| \frac{df^p(x_1)}{dx} \right| = \left| f'(x_p) \frac{df^{p-1}(x_1)}{dx} \right| = |f'(x_1) f'(x_2) \ldots f'(x_p)|$$

Numerically: Search in r to match this = 1.

Can find all periodic orbits with $p = 2^n$ starting from $r = 0$.

What else is there?

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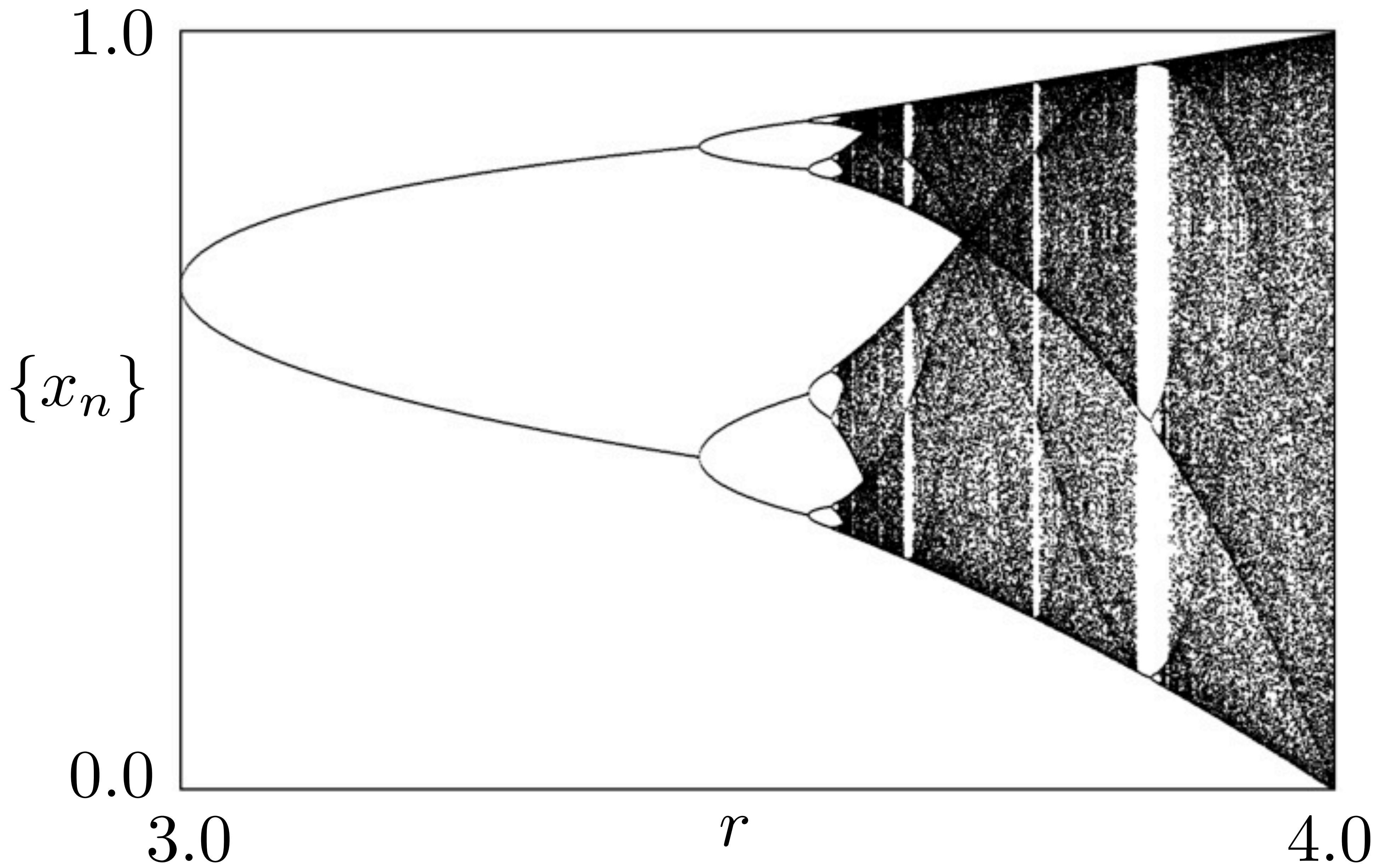
Bifurcation Theory of 1D Maps ...

Logistic map ...

Route to chaos via period-doubling cascade



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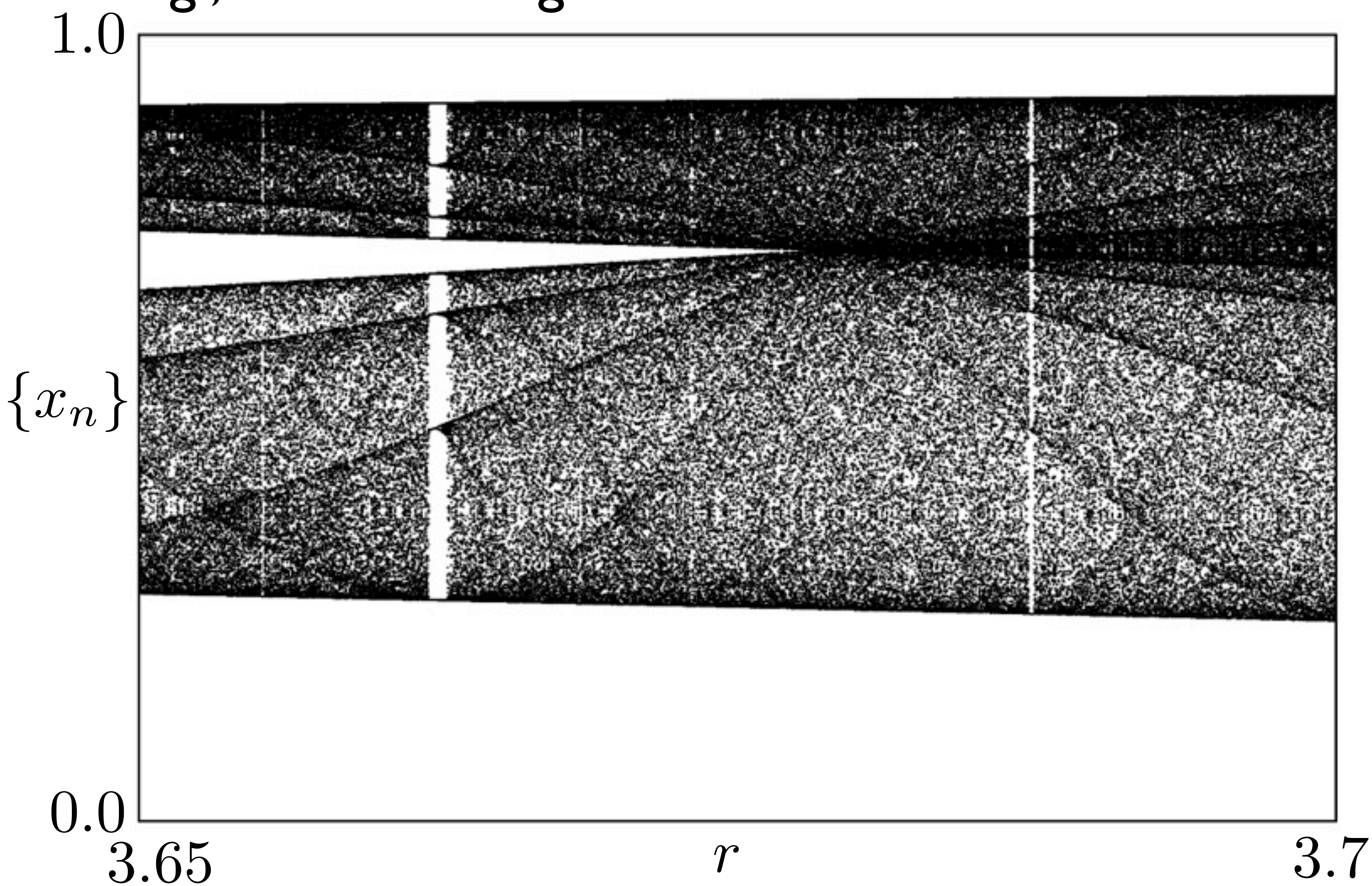
Band-merging (mirror of period-doubling):

E.g., 2 bands merge to 1 band



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Bifurcation Theory of 1D Maps ...

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What parameter values for band-merging?

Veils: Iterates $f^n(x_c)$ of map maximum $x_c = 1/2$

Upper bound on attractor: $f(x_c)$

Lower bound on attractor: $f^2(x_c)$

Two bands merge to one band: $f^k(x_c)$ becomes P-1

Specifically: $f^3(x_c) = f^4(x_c)$

Solve numerically: $r_{2B \rightarrow 1B} = 3.678\ldots$

Generally: 2^n bands merge to 2^{n-1} bands: $f^k(x_c)$ is period 2^{n-1}

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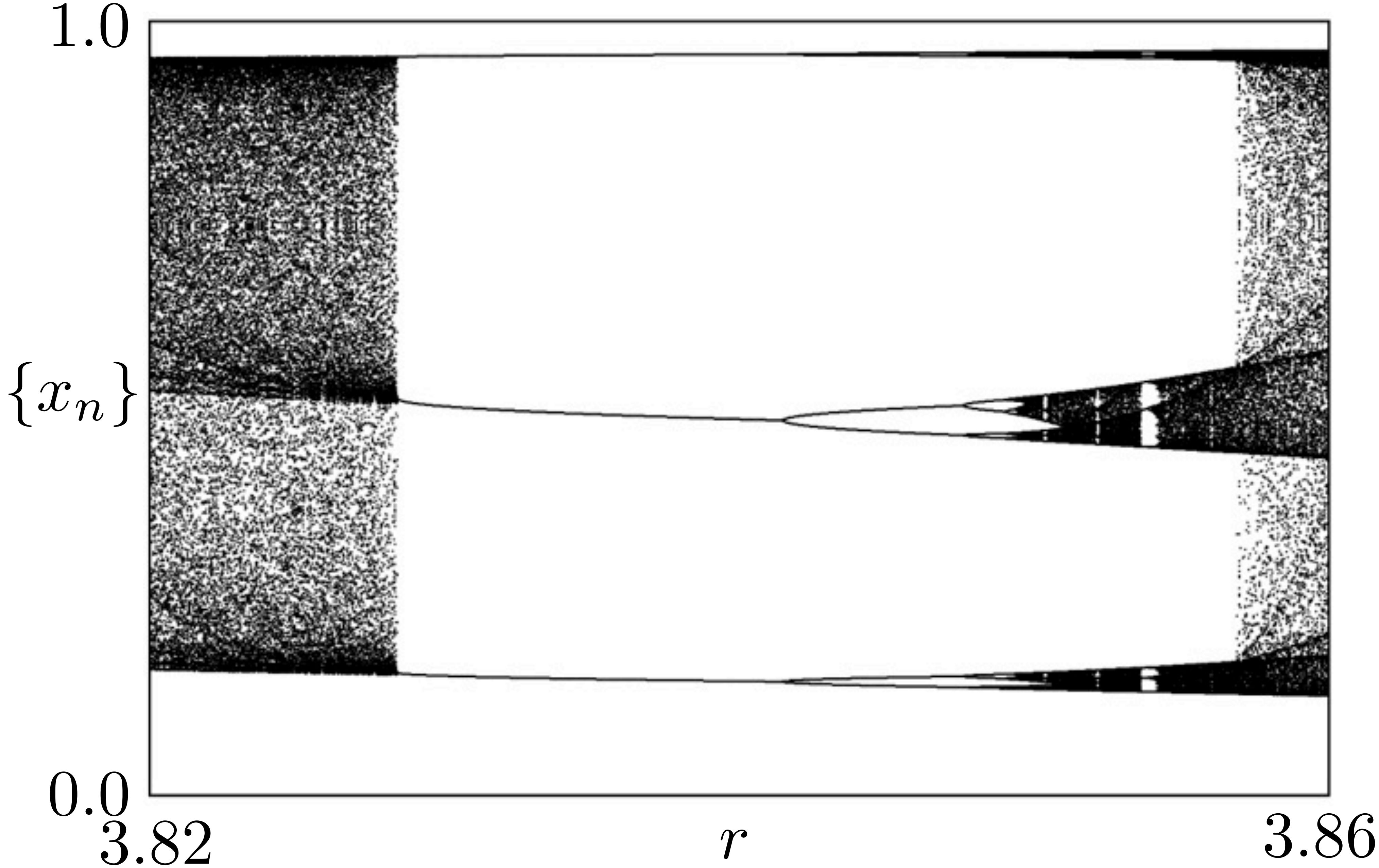
Bifurcation Theory of 1D Maps ...

Logistic map ...

Periodic windows



Entire period-doubling cascade inside window: $P = 3 \times 2^n$

Bifurcation Theory of 1D Maps ...

Logistic map ...

Periodic windows ...

How to locate:

Superstable periodic orbits: $x_i = x_c = \frac{1}{2}$

Why? $f'(x_c) = 0$

Period-3: $f^3(x_c) = x_c$

Solve numerically: $r_{P-3} = 3.83\ldots$

The Big, Big Picture (Bifurcations II) ...

Bifurcation Theory of 1D Maps ...

Logistic map ...

Simulation demos:

Animation as a function of parameter (ds)

B: nSteps = 10; r in [2.5,3.55]: nSteps = 10000; nTrans = 800; nIts = 500

B: nSteps = 10; r in [3.4,4]: nSteps = 10000; nTrans = 10000; nIts = 1000; color = 0

Bifurcation diagrams (bifn1d)

See usage

Bifurcations of 3D Flows:

- Simulation demos of Rössler: (ds)
  - Hopf bifurcation:
    - Fixed point to limit cycle: $c \in [0.1, 2.0]$
  - Period-doubling route: $c \in [1.0, 6.0]$

(2D Projection; #B; nSteps = 1 ; c is parameter 2; nSteps = 400; nTrans = 60000; nIts = 4000)

The Big, Big Picture (Bifurcations II) ...

Bifurcations of 3D Flows:

Simulation demo of driven van der Pol: (ds)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = A\sin(\omega t)$$

Limit cycle to torus
Limit cycle to chaos
Torus to chaos
Chaos to chaos

All of these in one sequence:

A = 3.0, w = 1.5, mu = 2.0:
2D proj; B; nSteps = 1 ; vary parameter 1 (A) in [0.1,5]; nSteps = 800; nTrans = 40000; nIts = 3000

The Big, Big Picture (Bifurcations II) ...

Next:

Chaotic mechanisms

Quantify the degree of chaos and unpredictability

Now:

A preview: Sounds of chaos

Rössler and Lorenz chaotic attractors

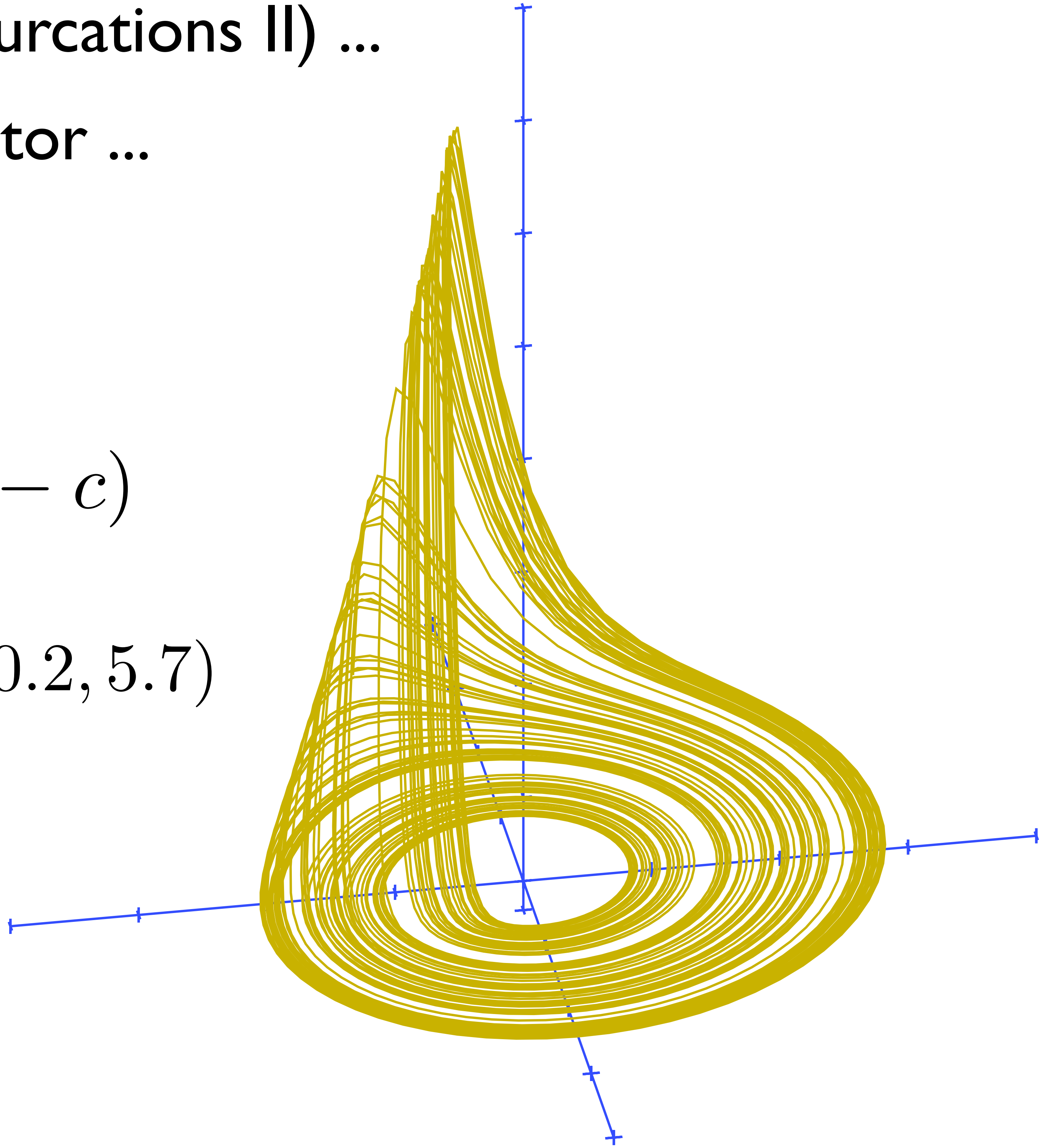
~/Programming/Audio/SoC

The Big, Big Picture (Bifurcations II) ...

Rössler chaotic attractor ...

$$\dot{x} = -y - z$$
$$\dot{y} = x + ay$$
$$\dot{z} = b + z(x - c)$$

$$(a, b, c) = (0.2, 0.2, 5.7)$$



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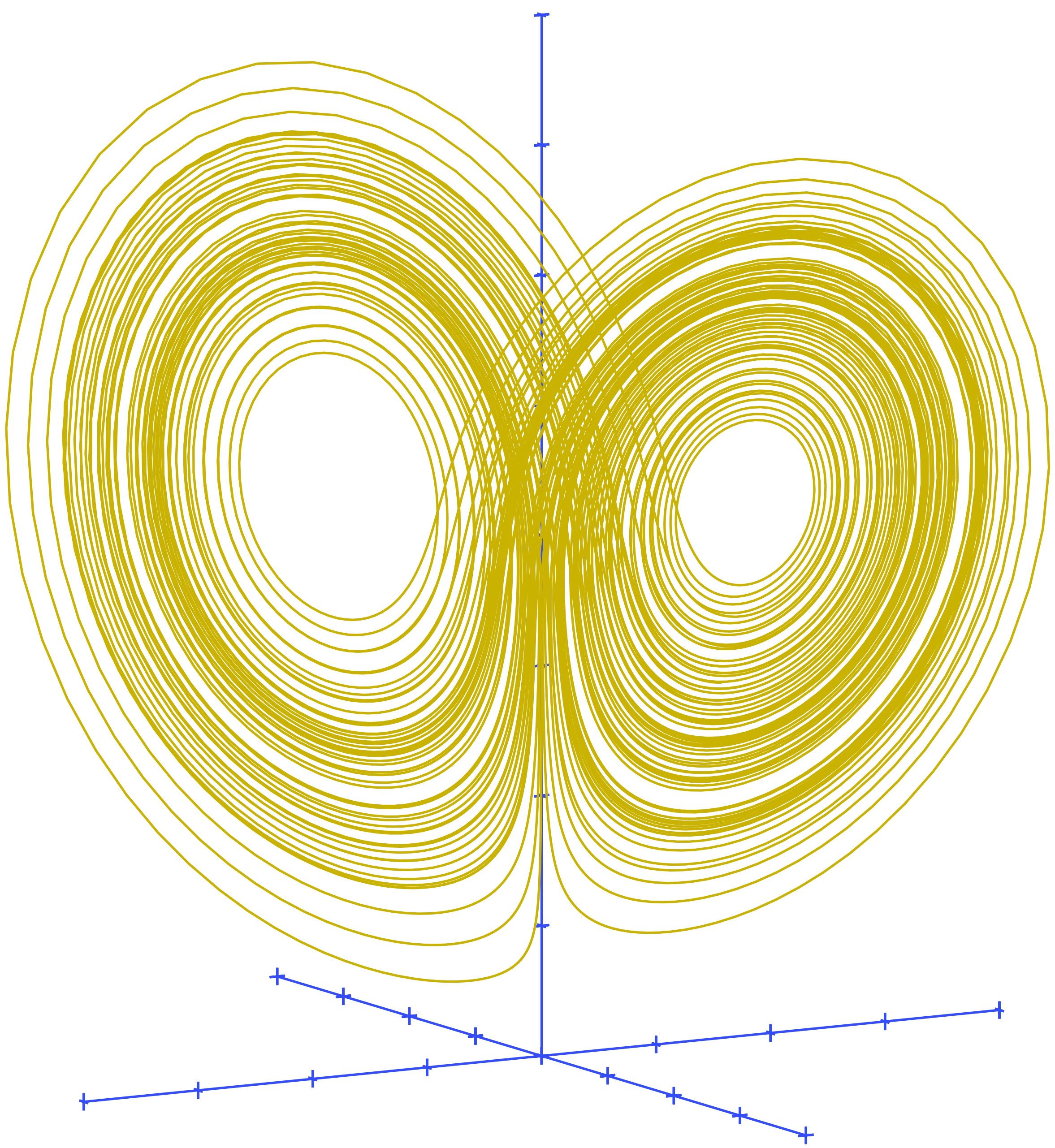
The Big, Big Picture (Bifurcations II) ...

Lorenz chaotic attractor ...

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz \end{aligned}$$

$$(\sigma, r, b) = (10, 8/3, 28)$$



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The Big, Big Picture (Bifurcations II) ...

Reading for next lecture:

NDAC, Sec. 12.0-12.3, 9.3, and 10.5.