Example Dynamical Systems

Reading for this lecture:

NDAC, Sec. 6.0-6.4, 7.0-7.3, & 9.0-9.4

Example Dynamical Systems ...

1D Flows: Fixed Points

model of static equilibrium

1D Flow: $x \in \mathbb{R}$

$$\dot{x} = F(x)$$

Fixed Points:

$x^* \in \mathbb{R}$ such that

$$\dot{x}\,|_{x^*} = 0$$

or

$$F(x^*) = 0$$

1D Flows: Fixed Points ...

Stability: What is linearized system at x ?

Investigate evolution of perturbations: $x' = x + \delta x \qquad |\delta x| \ll 1$

$$\dot{x} = F(x)$$

Local Flow: $\delta \dot{x} = \left. \frac{dF}{dx} \right|_{x(t)} \delta x$

Local Linear System: $\delta \dot{x} = \lambda \, \delta x$

Solution: $\delta x(t) \propto e^{\lambda t} \delta x(0)$

Example Dynamical Systems ...

1D Flows ...

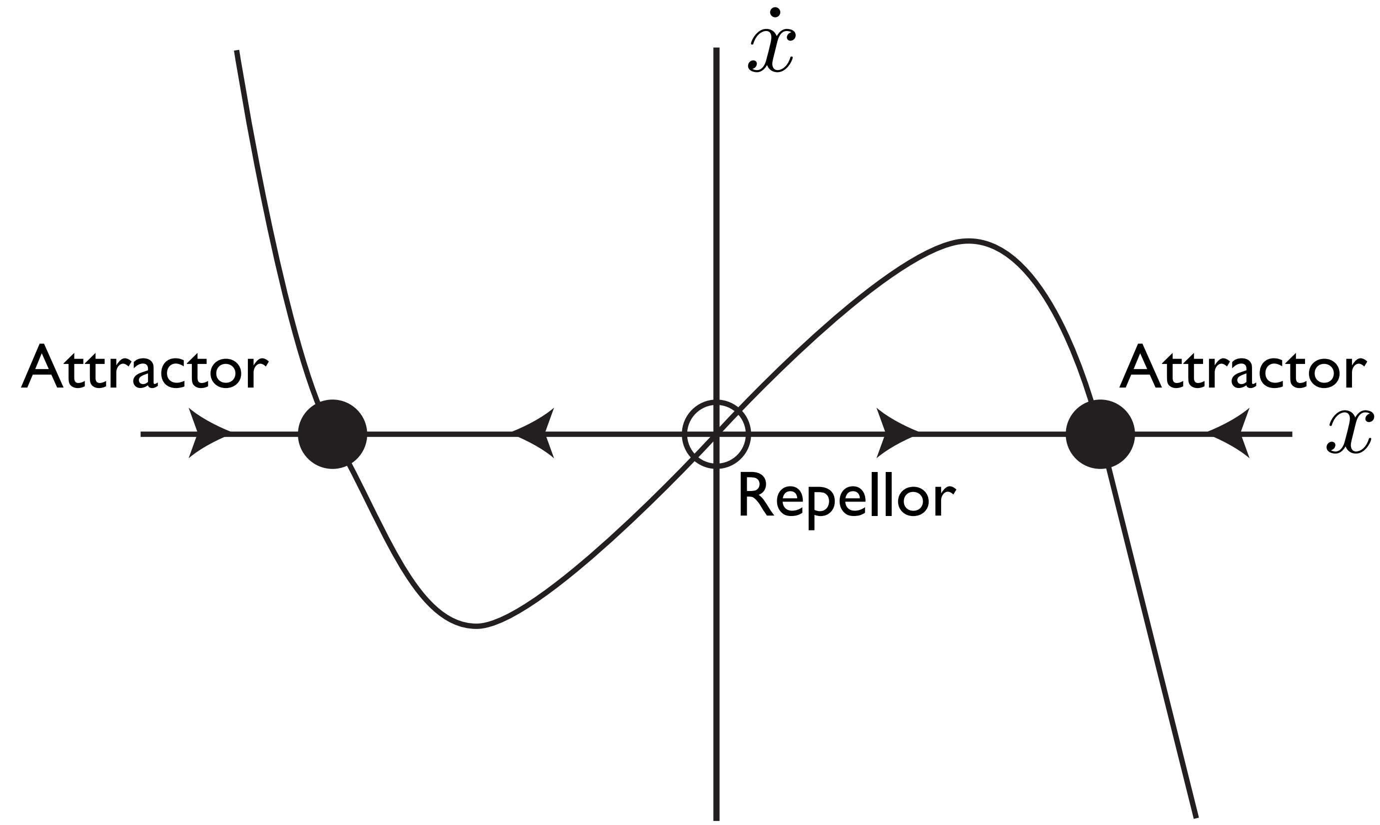
Stability Classification of Fixed Points:

Slope λ of $F(x)$ at x :

1. Stable: $\lambda < 0$

2. Unstable: $\lambda > 0$

3. Neutral: $\lambda = 0$



Example Dynamical Systems ...

Linear algebra review:

(See, for example, *NDAC*, Chapter 5)

$n \times n$ matrix: A

Determinant: $\mathrm{Det}(A)$

Trace: $\mathrm{Tr}(A)$

For example:

2×2 matrix: $A = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$

$$\mathrm{Det}(A) = ad - bc$$

$$\mathrm{Tr}(A) = a + d$$

Example Dynamical Systems ...

Linear algebra review ...

(Left) eigensytem: $\lambda_i \vec{v}_i = \vec{v}_i \cdot A$

(Left) eigenvalues: $\{\lambda_i \in \mathbb{C} : i = 1, \ldots, n\}$ Scaling factors

(Left) eigenvectors: $\{\vec{v}_i \in \mathbb{R}^n : i = 1, \ldots, n\}$ Eigen-directions that are invariant

Example Dynamical Systems ...

Linear algebra review ...

(Right) eigensytem: $\lambda_i \vec{v}_i = A \cdot \vec{v}_i$

(Right) eigenvalues: $\{\lambda_i \in \mathbb{C} : i = 1, \ldots, n\}$ Scaling factors

(Right) eigenvectors: $\{\vec{v}_i \in \mathbb{R}^n : i = 1, \ldots, n\}$ Eigen-directions that are invariant

Example Dynamical Systems ...

Linear algebra review ...

Determinant: $$\mathrm{Det}(A) = \prod_{i=1}^{n} \lambda_i$$

Trace: $$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$$

Example Dynamical Systems ...

2D Flows: Fixed Points

model of static equilibrium

2D Flow: $\vec{x} \in \mathbf{R}^2$

$$\dot{\vec{x}} = \vec{F}(\vec{x})$$

or

$$\begin{aligned} \vec{x} &= (x, y) \\ \vec{F} &= (f, g) \end{aligned}$$

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

Fixed Points:

(x^*, y^*) such that

$$\dot{\vec{x}}|_{(x^*, y^*)} = (0, 0)$$

or

$$0 = f(x^*, y^*)$$

$$0 = g(x^*, y^*)$$

Example Dynamical Systems ...

2D Flows: Fixed Points ...

model of static equilibrium

Stability: What is linearized system at $\vec{x}$?

Investigate evolution of perturbations: $\vec{x}' = \vec{x} + \delta\vec{x}$ $\quad |\delta\vec{x}| \ll 1$

$$\dot{\vec{x}} = \vec{F}(\vec{x})$$

Local Flow: $$\delta\dot{\vec{x}} = \left.\frac{\partial\vec{F}}{\partial\vec{x}}\right|_{\vec{x}(t)} \cdot \delta\vec{x}$$

Initial conditions: $x(0)$ $\delta x(0)$

2D Flows: Fixed Points ...

Local Linear System: $\delta\dot{\vec{x}} = A \cdot \delta\vec{x}$

Jacobian: $A = \frac{\partial \vec{F}}{\partial \vec{x}} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$

Solution:

$$\delta\vec{x}(t) \propto e^{At}\delta\vec{x}(0)$$

2D Flows: Fixed Points (an aside) ...

Solve linear ODEs: Find $x(t)$ given

$$x(0)$$

$$\dot{x} = Ax$$

Eigenvalues and eigenvectors: $\lambda_j \;\; v_j$

$$Av_j = \lambda_j, \; j = 1, \ldots, n$$

Solution:

$$x(t) = \sum_{j=1} \alpha_j e^{\lambda_j t} v_j$$

where calculate α_j :

$$x(0) = \sum_{j=1} \alpha_j v_j$$

Example Dynamical Systems ...
2D Flows: Fixed Points ...

Linear system: $\delta\dot{\vec{x}} = A \cdot \delta\vec{x}$

If $\delta\vec{x}$ aligns with an eigen-direction, then $A \sim \begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$

and

$$\begin{aligned} \delta\dot{x}_1 &= \lambda_1 \, \delta x_1 \\ \delta\dot{x}_2 &= \lambda_2 \, \delta x_2 \\ &\vdots \end{aligned}$$

so that

$$\begin{aligned} \delta x_1(t) &= e^{\lambda_1 t} \cdot \delta x_1(0) \\ \delta x_2(t) &= e^{\lambda_2 t} \cdot \delta x_2(0) \\ &\vdots \end{aligned}$$

$\lambda_i > 0$ Growth

$\lambda_i < 0$ Decay

Example Dynamical Systems ...

2D Flows: Fixed Points ...

Simple harmonic oscillator:

$$\ddot{x} + x = 0$$

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}$$

Fixed point at the origin: $(x, y) = (0, 0)$

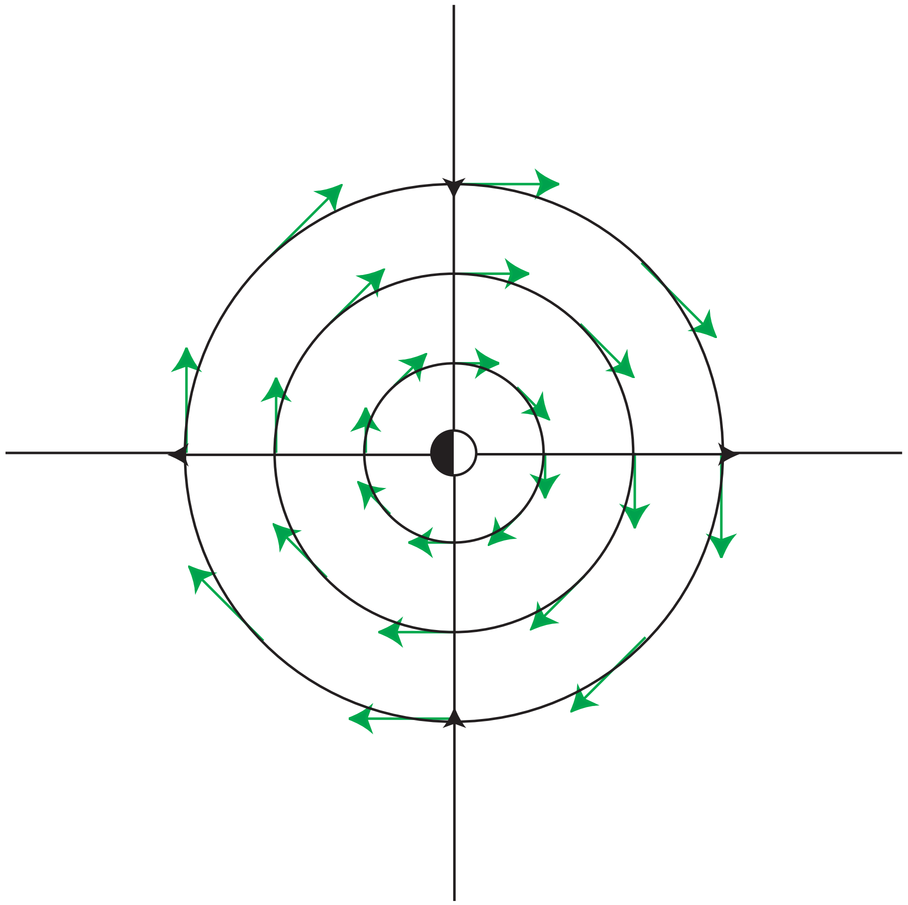
Jacobian: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathrm{Det}(A) = 1$$

$$\mathrm{Tr}(A) = 0$$

$$\lambda_1 = i$$

$$\lambda_2 = -i$$



Example Dynamical Systems ...

2D Flows: Fixed Points ...

Damped harmonic oscillator:

$$\ddot{x} + \gamma \dot{x} + x = 0 \qquad \text{or} \qquad \begin{aligned} \dot{x} &= y \\ \dot{y} &= -x - \gamma y \end{aligned}$$

Fixed point at the origin: $(x, y) = (0, 0)$

Jacobian: $A = \begin{pmatrix} 0 & 1 \\ -1 & -\gamma \end{pmatrix}$

$\mathrm{Det}(A) = 1$ $\qquad$ $\gamma > 0$ Damped

$\mathrm{Tr}(A) = -\gamma$ $\qquad$ $\gamma < 0$ Unstable

Example Dynamical Systems ...

2D Flows: Fixed Points ...

Damped harmonic oscillator ...

Eigenvalues: $$\lambda_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4}}{2}$$

$$\lambda_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4}}{2}$$

Real when: $|\gamma| \geq 2$ Critically damped

Complex when: $|\gamma| < 2$ Under-damped: spiral

Example Dynamical Systems ...

2D Flows ...

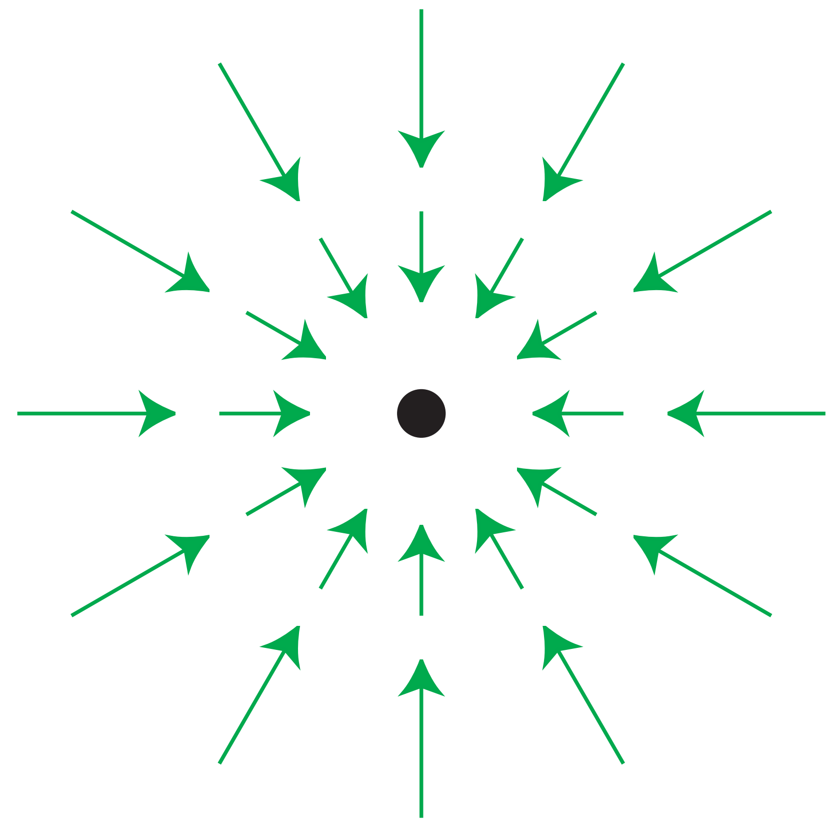
Stability Classification of Fixed Points:

Eigenvalues of Jacobian A at $\vec{x}$: $\lambda_1 \ \& \ \lambda_2 \ \in \ \mathbf{C}$

(Review: *NDAC*, Chapter 5)

Stable fixed point (aka sink, attractor):

$$\Re(\lambda_1), \Re(\lambda_2) < 0$$

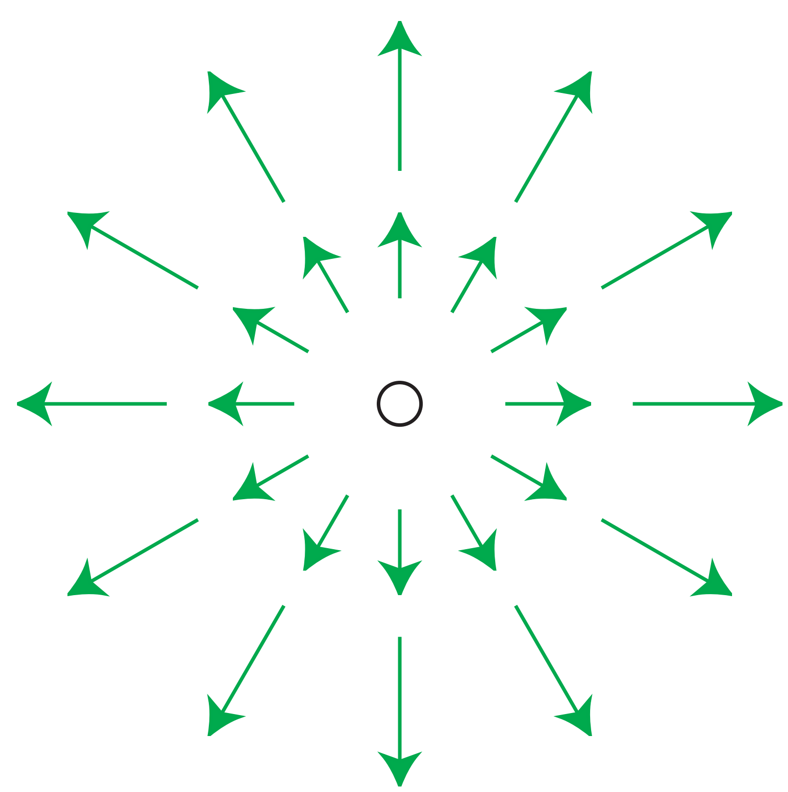


Example Dynamical Systems ...
 2D Flows ...
  Stability Classification of Fixed Points ...

Eigenvalues of Jacobian A at $\vec{x}: \lambda_1 \ \& \ \lambda_2 \ \in \ \mathbf{C}$

Unstable fixed point (aka source, repellor):

$$\Re(\lambda_1), \Re(\lambda_2) > 0$$



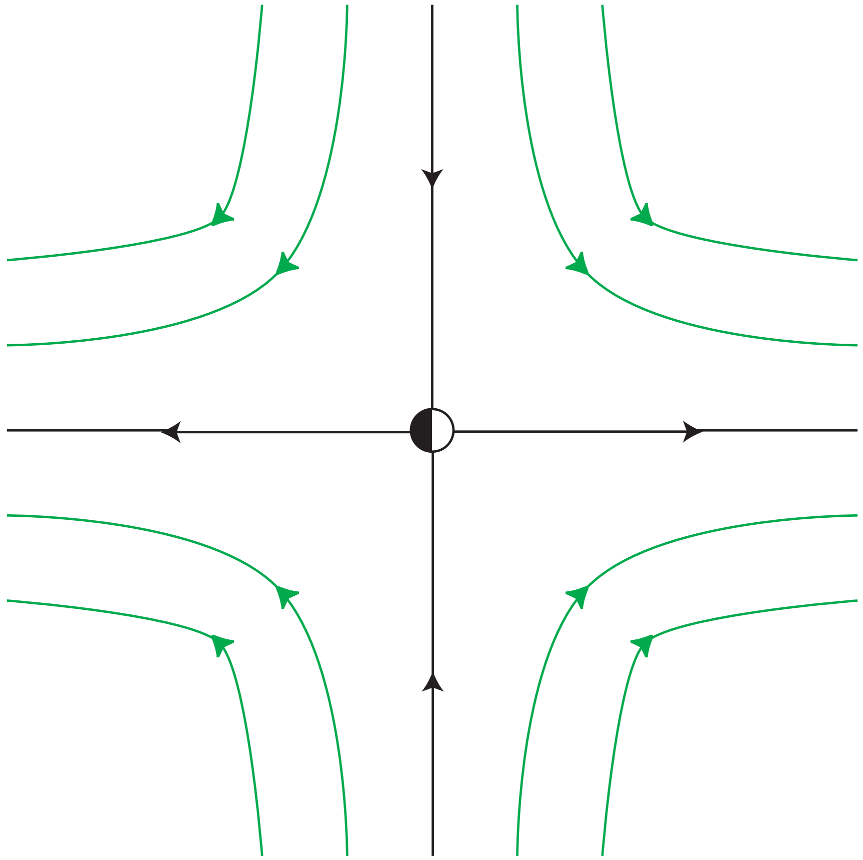
Lecture 3: Nonlinear Physics, Physics 150/250 (Spring 2010); Jim Crutchfield

Example Dynamical Systems ...
2D Flows ...
Stability Classification of Fixed Points:

Eigenvalues of Jacobian at $\vec{x}$: $\lambda_1 \;\&\; \lambda_2 \;\in\; \mathbf{C}$

Saddle fixed point (mixed stability):

$$\Re(\lambda_1) > 0 \;\&\; \Re(\lambda_2) < 0$$

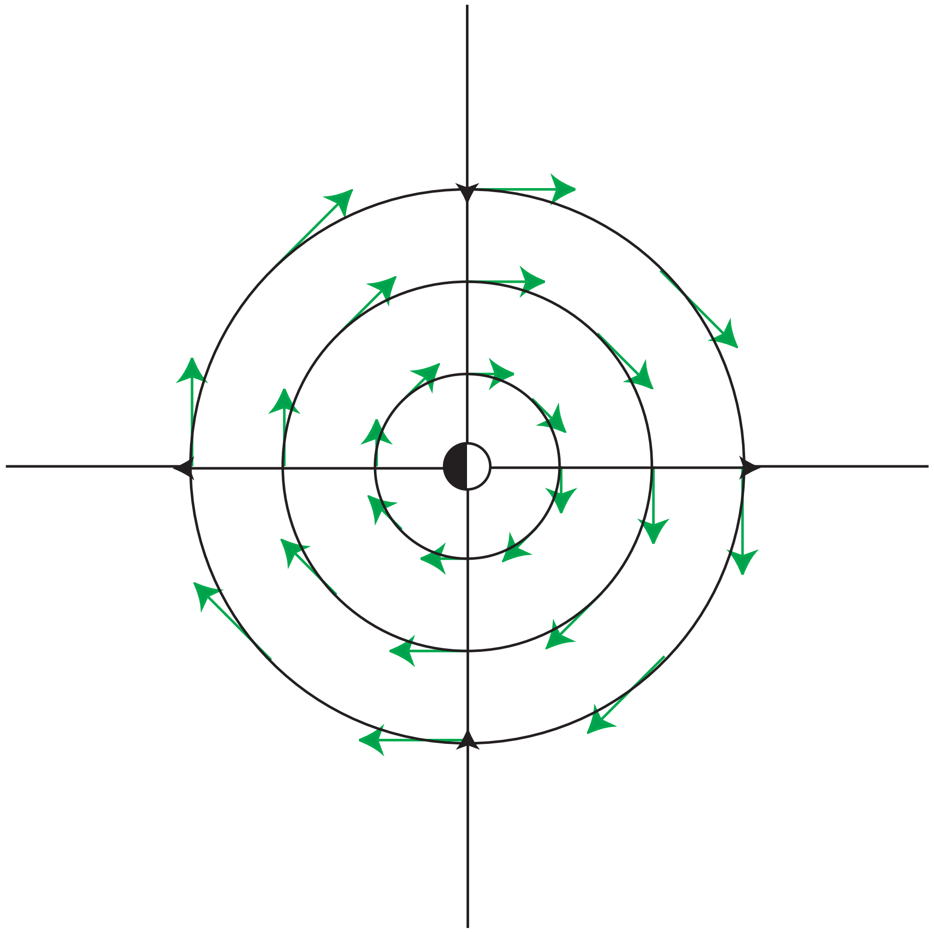


Example Dynamical Systems ...
2D Flows ...
Stability Classification of Fixed Points:

Eigenvalues of Jacobian at $\vec{x}$: $\lambda_1 \;\&\; \lambda_2 \in \mathbf{C}$

Center:

$$\Re(\lambda_1) = \Re(\lambda_2) = 0$$



Example Dynamical Systems ...

2D Flows ...

Stability Classification of Fixed Points ...

Class of submanifolds: $\mathrm{Det}(A) = \lambda_1 \cdot \lambda_2$

$\mathrm{Det}(A) < 0 : \lambda_1, \lambda_2 \in \mathbf{R},\ \lambda_1 > 0 \Rightarrow \lambda_2 < 0$ Saddles

$\mathrm{Det}(A) > 0 :$ Stability $\mathrm{Tr}(A) = \lambda_1 + \lambda_2$

Stable: $\mathrm{Tr}(A) < 0$

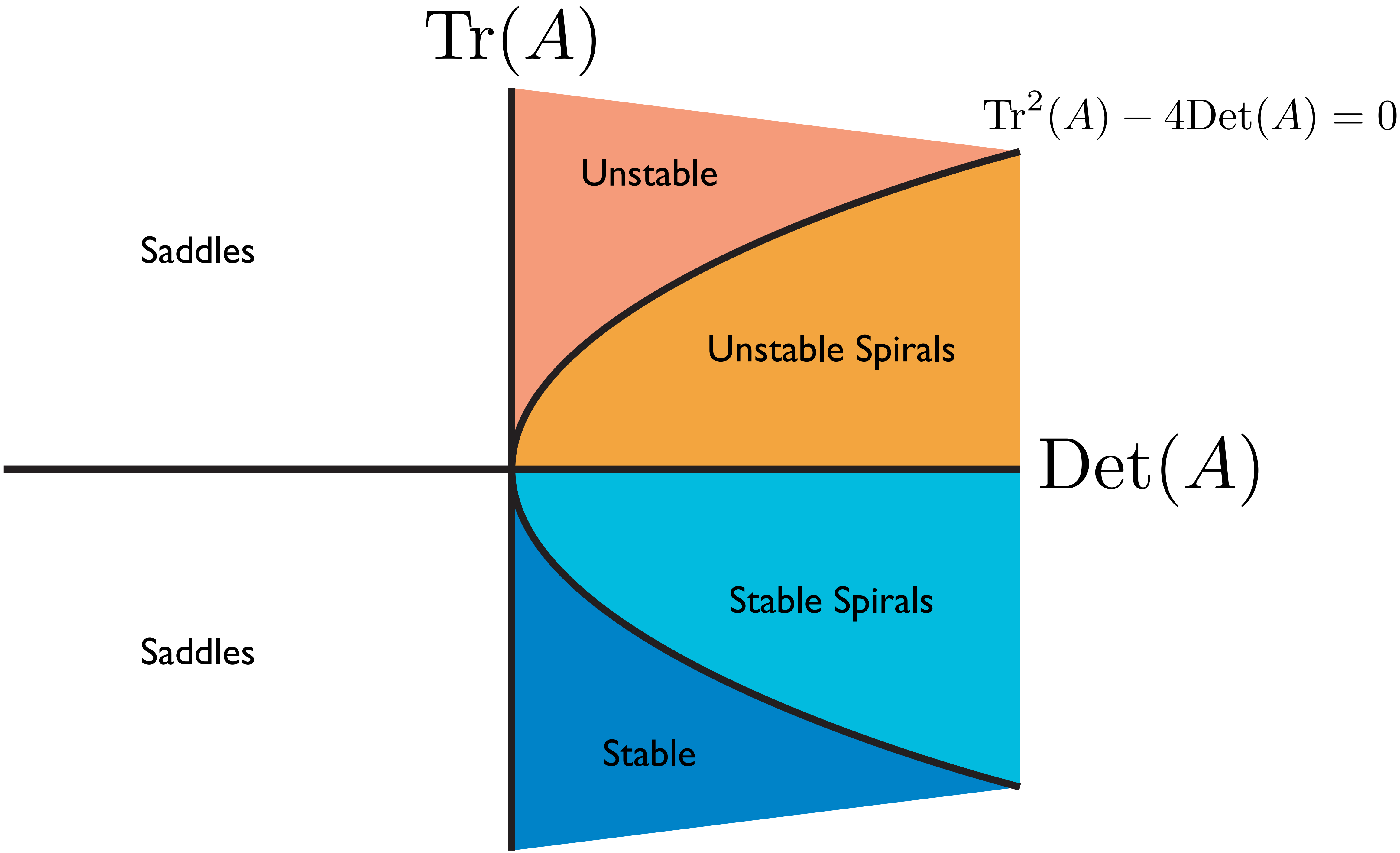
Unstable: $\mathrm{Tr}(A) > 0$

Marginal: $\mathrm{Tr}(A) = 0$

Example Dynamical Systems ...

2D Flows ...

Stability Classification of Fixed Points ...



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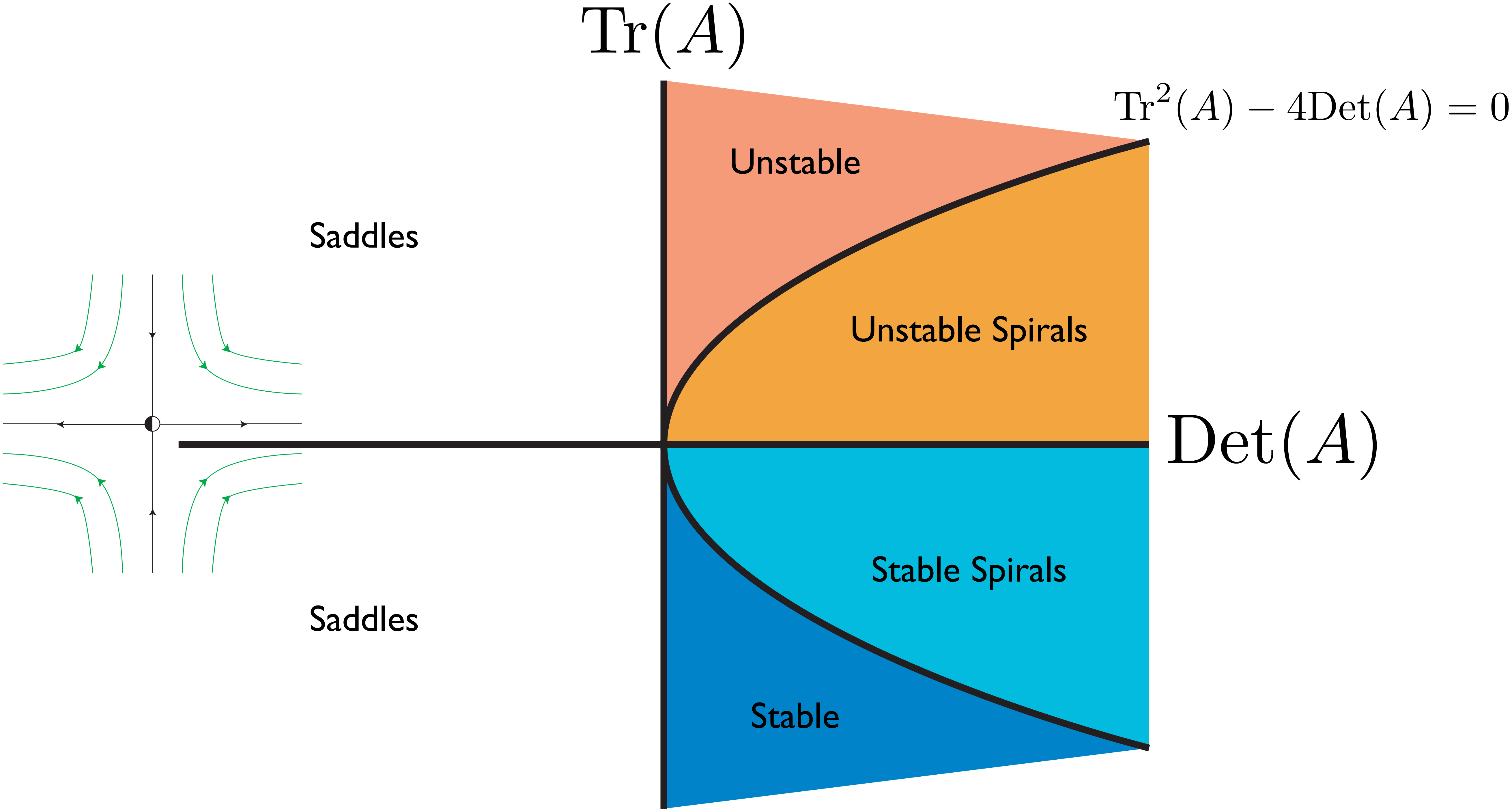
Example Dynamical Systems ...

2D Flows ...

Stability Classification of Fixed Points ...



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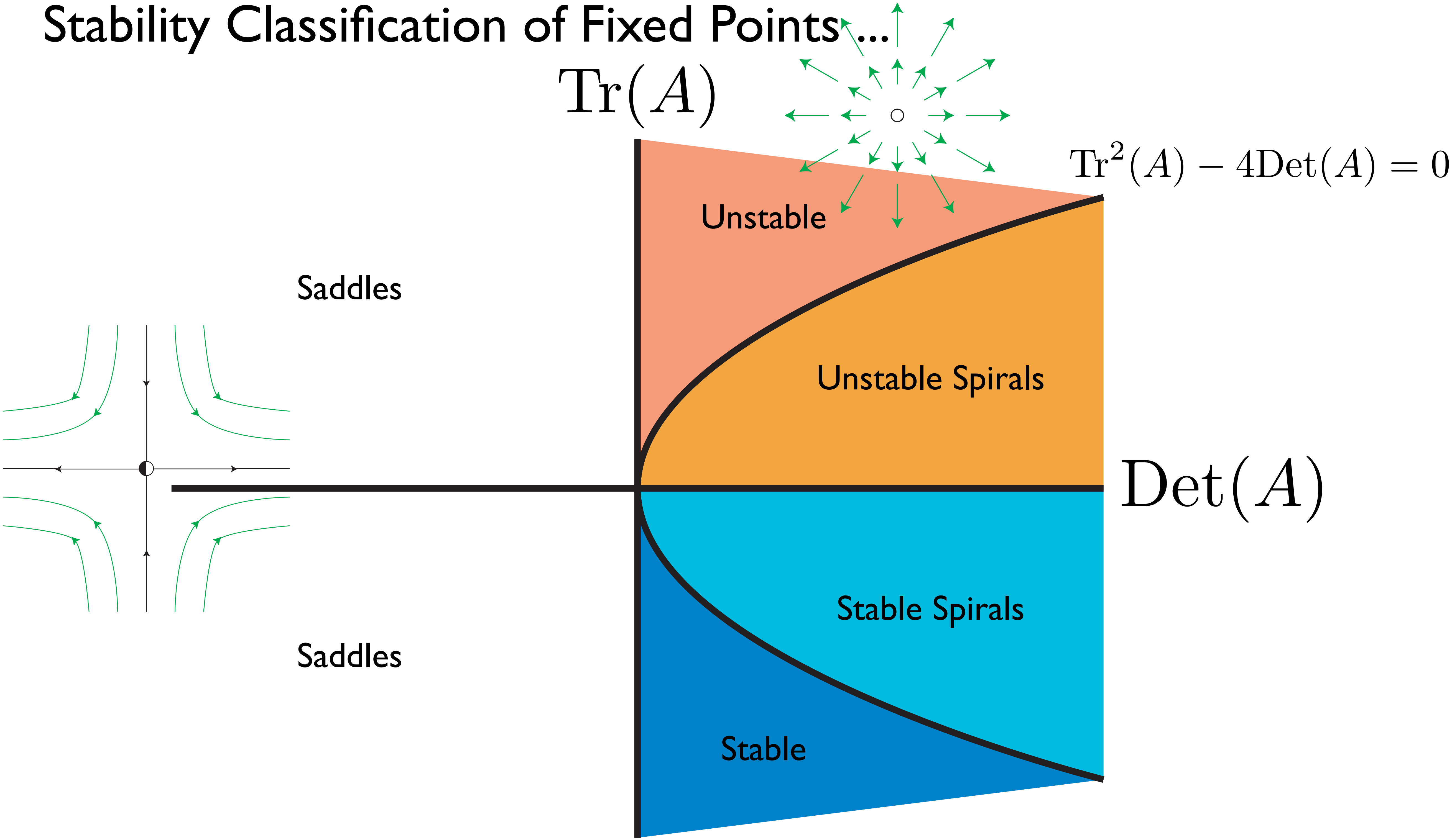
Example Dynamical Systems ...

2D Flows ...

Stability Classification of Fixed Points ...



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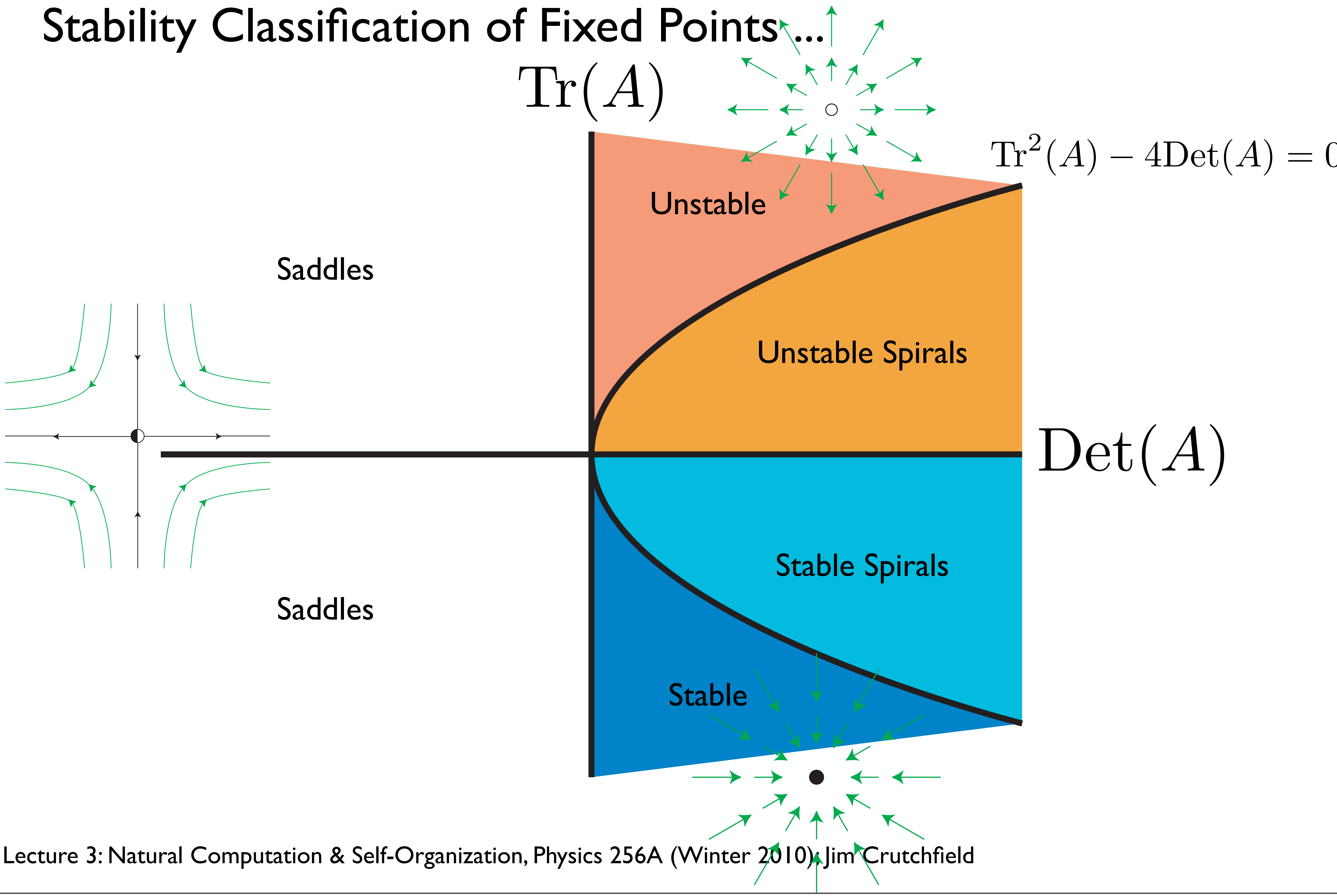
Example Dynamical Systems ...

2D Flows ...

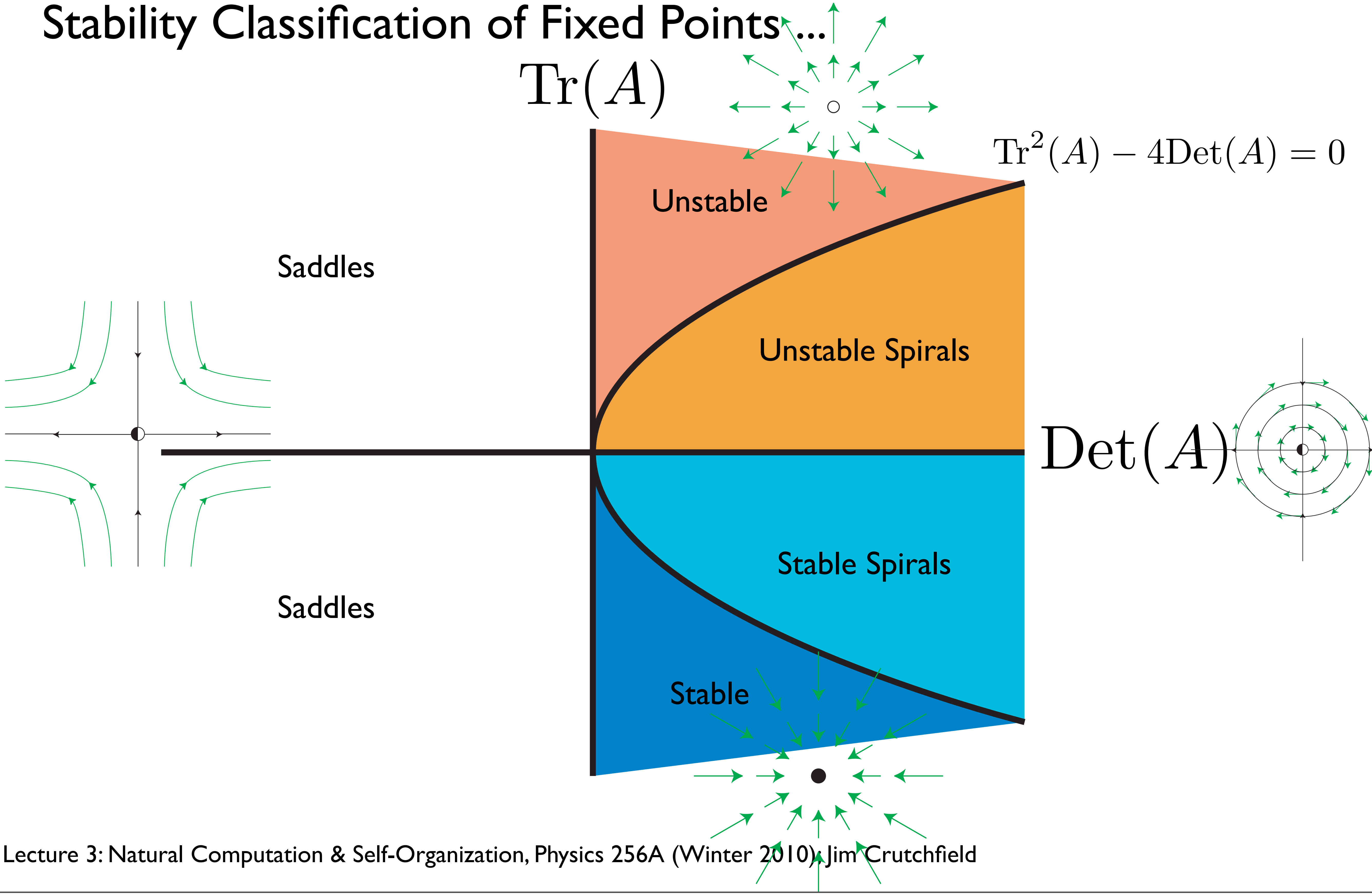
Stability Classification of Fixed Points ...



Example Dynamical Systems ...

2D Flows ...

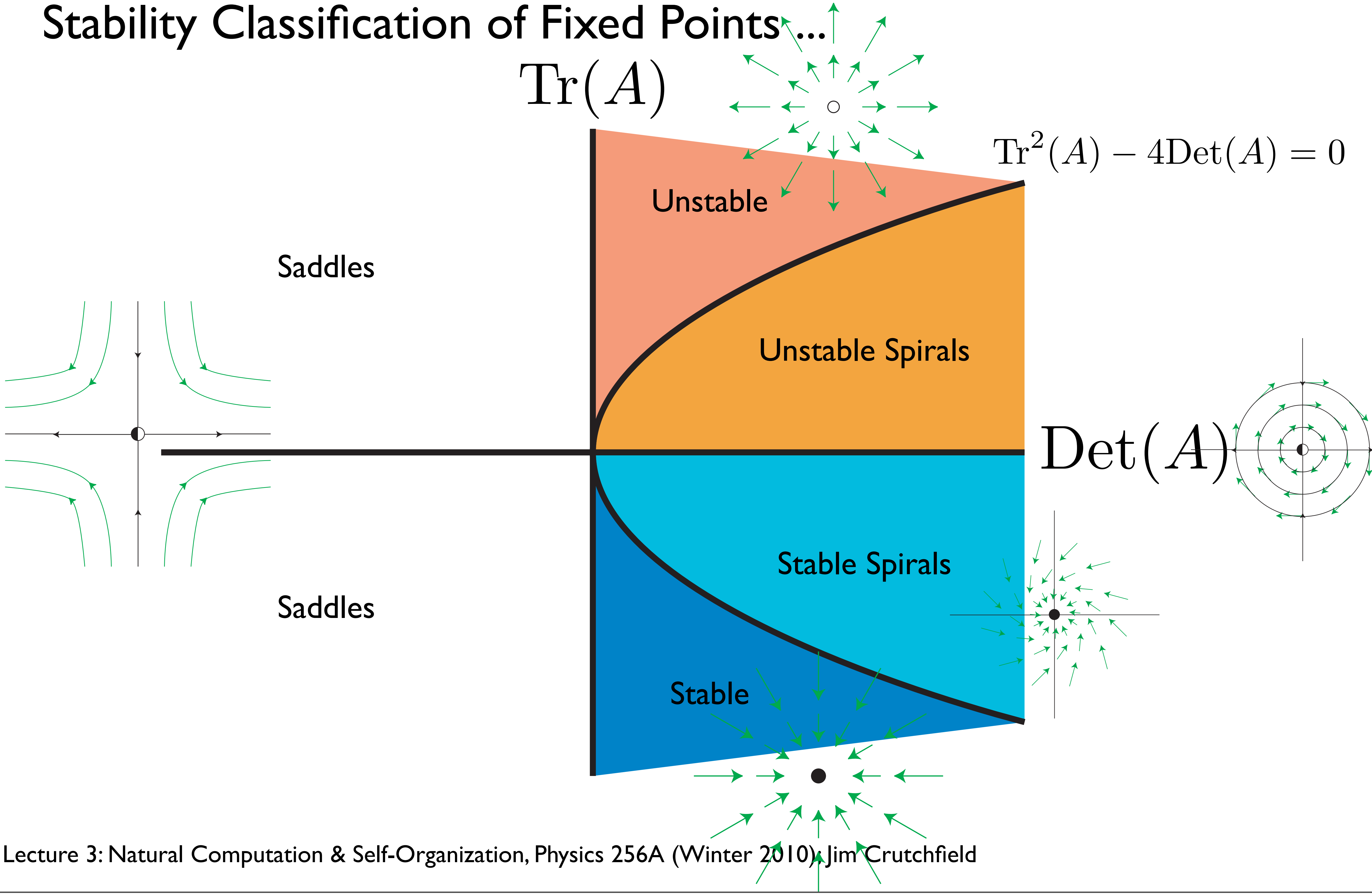
Stability Classification of Fixed Points ...



Example Dynamical Systems ...

2D Flows ...

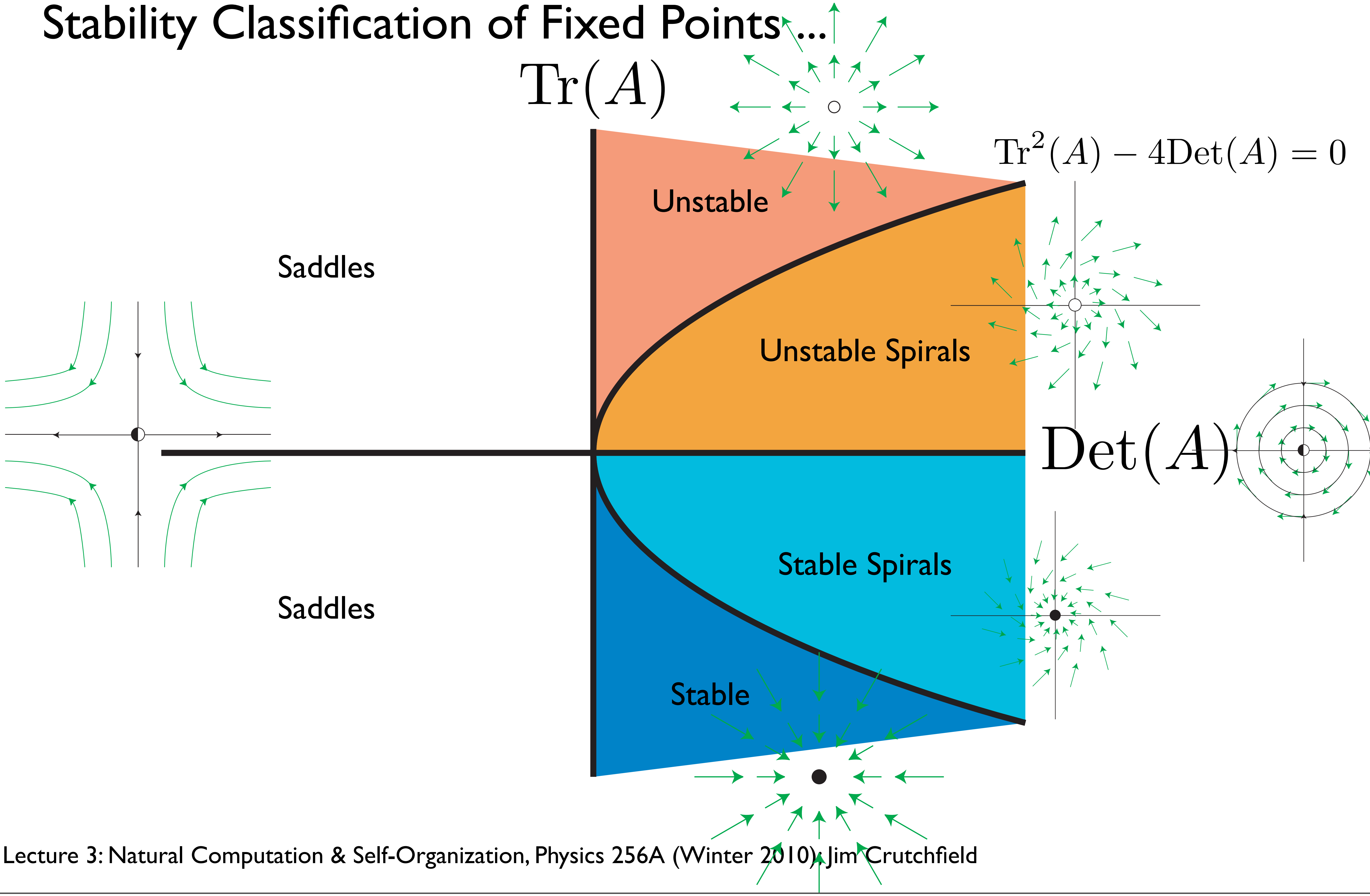
Stability Classification of Fixed Points ...



Example Dynamical Systems ...

2D Flows ...

Stability Classification of Fixed Points ...



Example Dynamical Systems ...

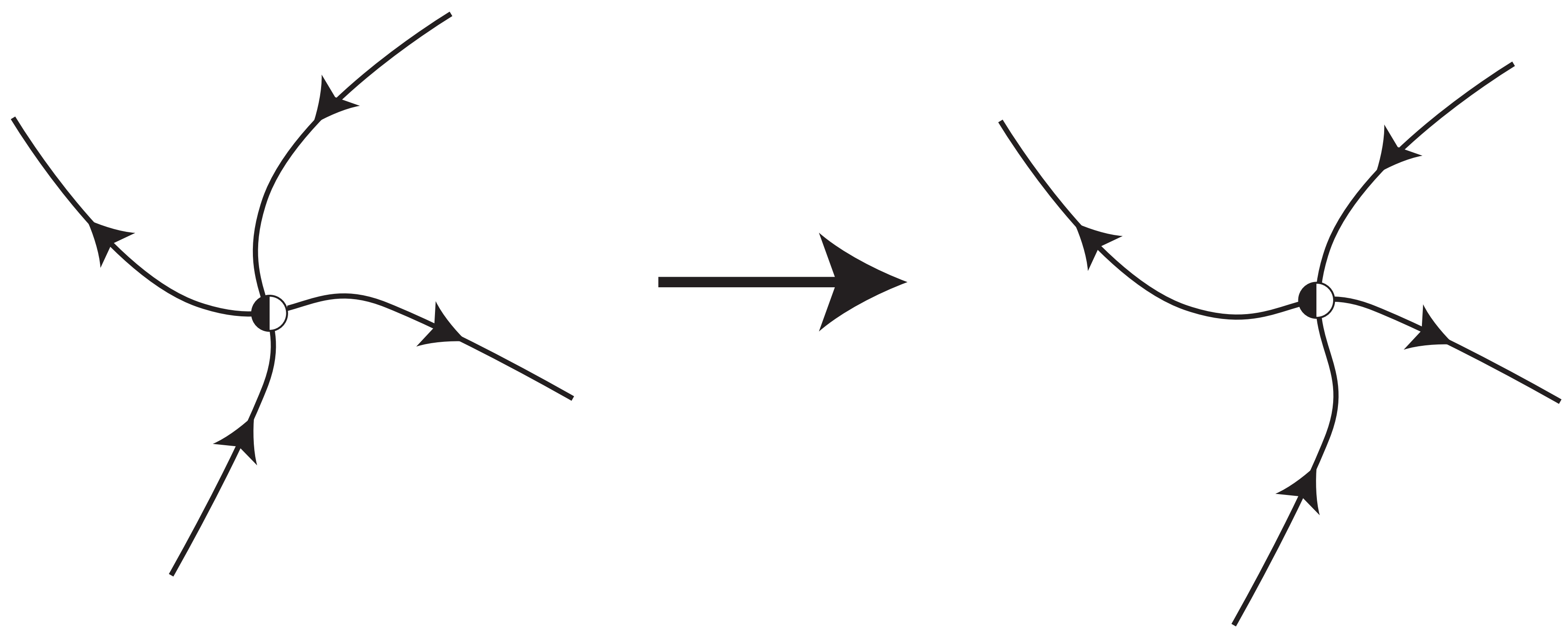
2D Flows ...

Stability Classification of Fixed Points ...

Hyperbolic intersection of W^s and W^u:

Robust, if $\Re(\lambda_i) \neq 0, \forall i$

Saddle fixed point persists under perturbation



Example Dynamical Systems ...

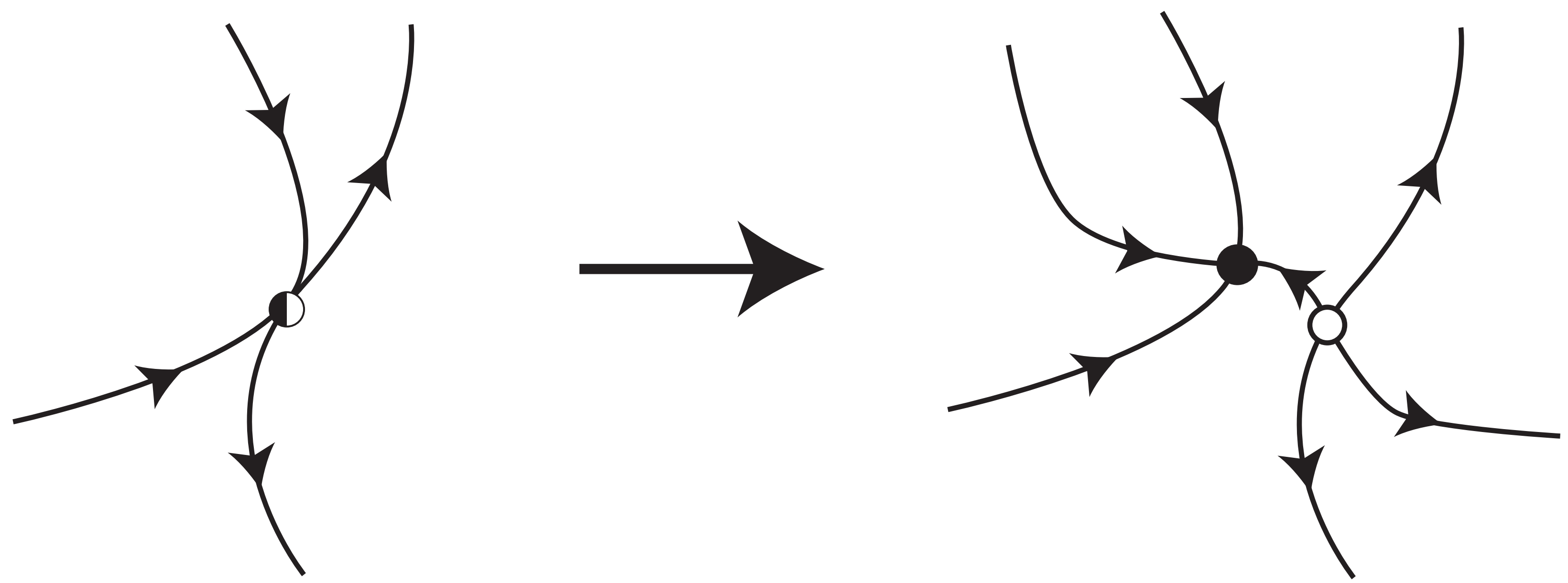
2D Flows ...

Stability Classification of Fixed Points ...

Non-hyperbolic intersection of W^s and W^u:

Fragile

Saddle fixed point changes structure under perturbation



Example Dynamical Systems ...

2D Flows: Limit Cycles

isolated, closed trajectory:

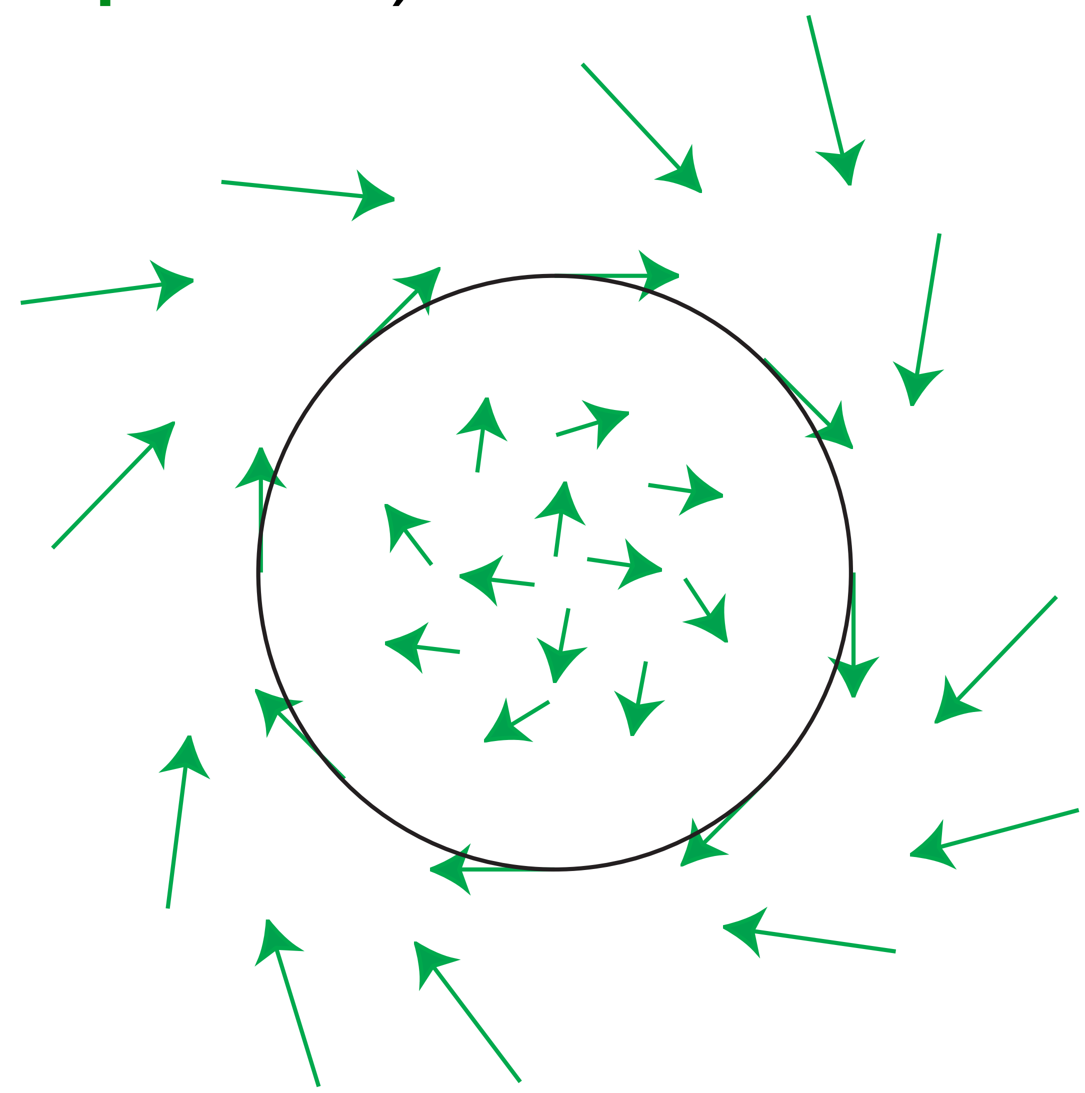
a periodic orbit: $\vec{x}(t) = \vec{x}(t+p), \text{ for all } t$

(p is the period)

model of stable oscillation

this is a new behavior type

not possible in 1D flows



Stable limit cycle

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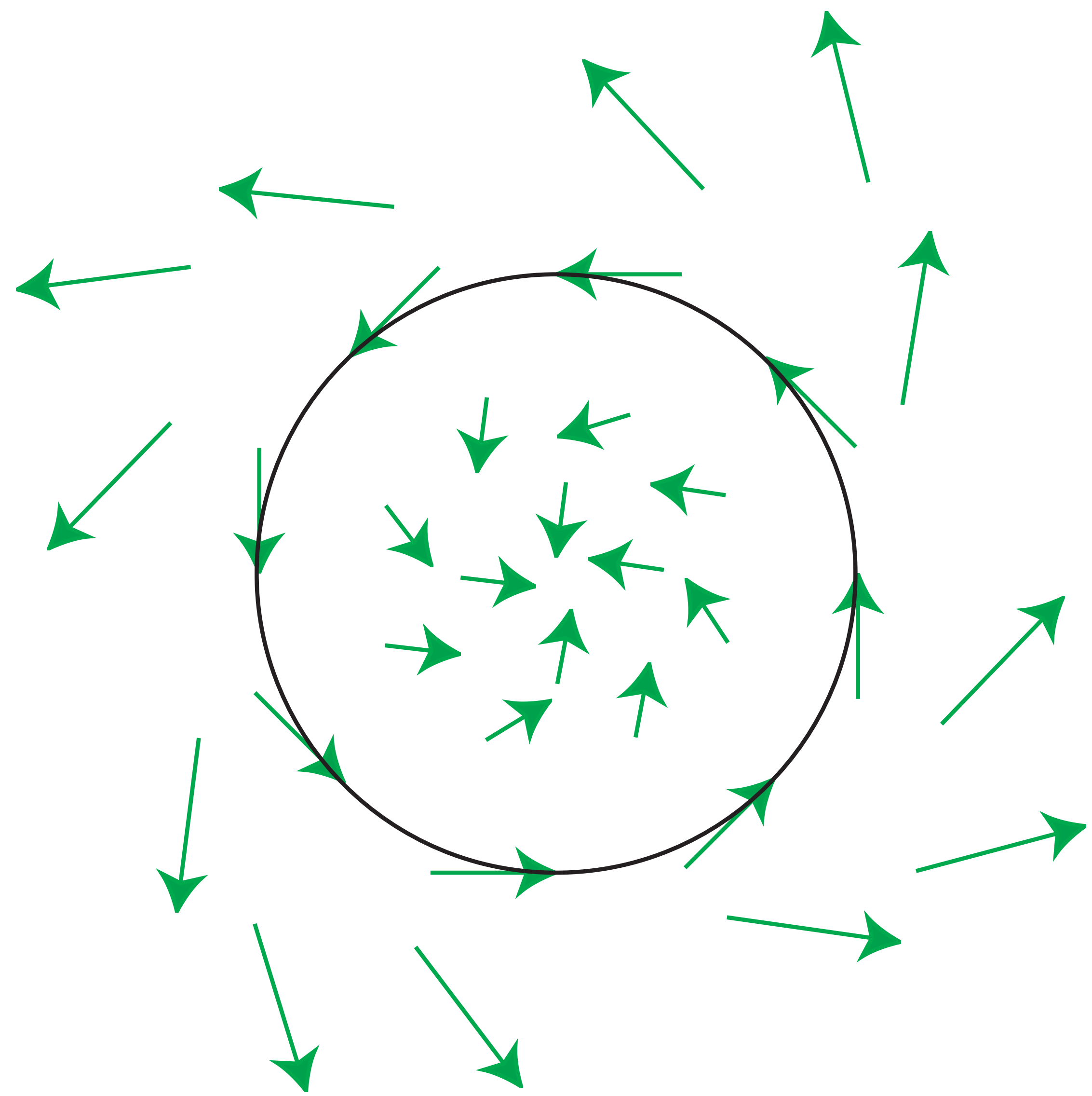
Example Dynamical Systems ...

2D Flows: Limit Cycles ...

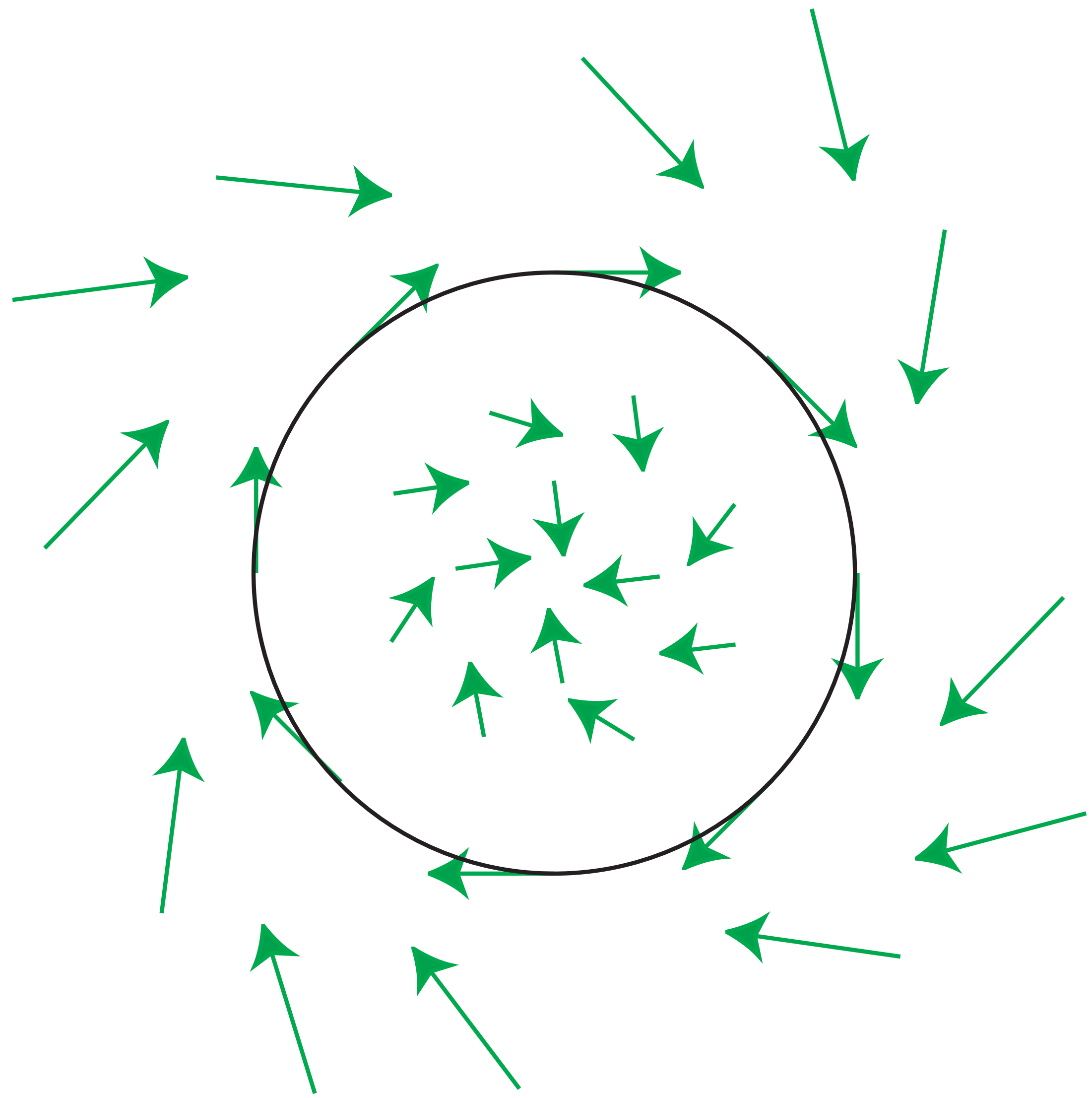
Unstable cycle



Example Dynamical Systems ...

2D Flows: Limit Cycles ...

Saddle cycle



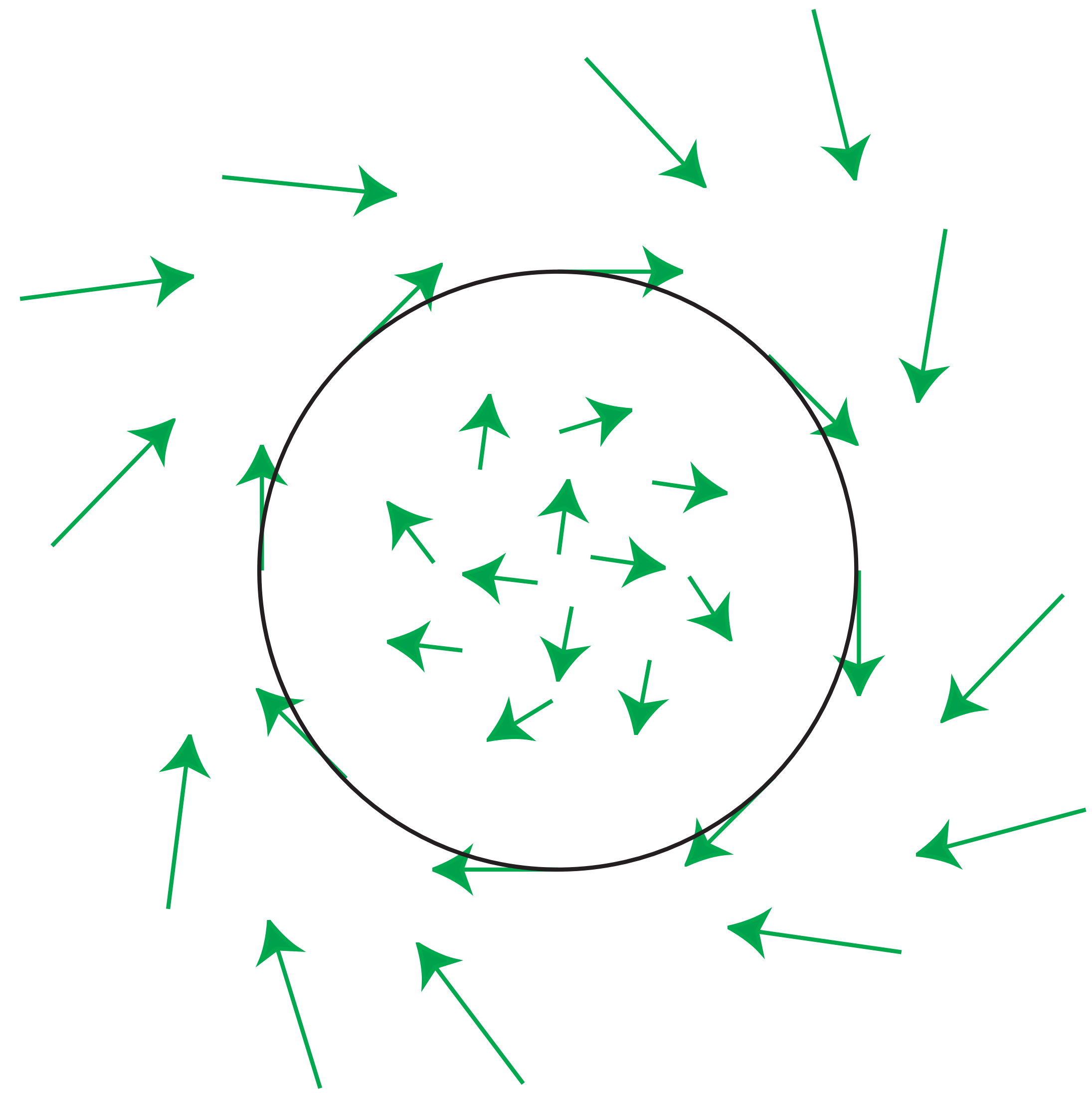
Example Dynamical Systems ...

2D Flows ...

Limit Cycle Examples

Easy in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$



Example Dynamical Systems ...

2D Flows ...

Limit Cycle Examples ...

Van der Pol Equations:

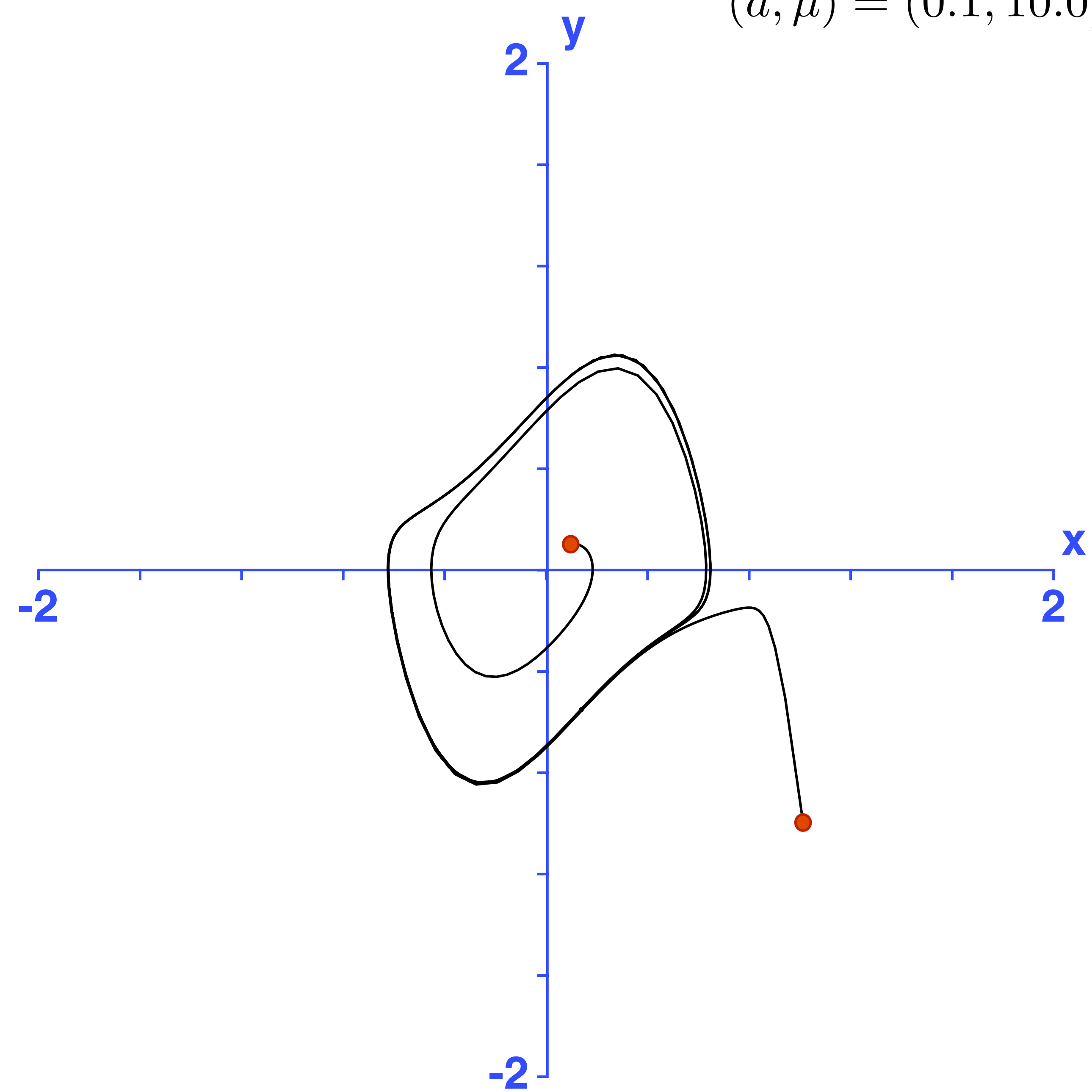
$$\ddot{x} + \mu(x^2 - a)\dot{x} + x = 0$$

or

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + \mu y(a - x^2) \end{aligned}$$

Nonlinear damping changes sign:
- Small oscillation ($x < \sqrt{a}$): Growth
- Large oscillation ($x > \sqrt{a}$): Damped

$(a, \mu) = (0.1, 10.0)$



Example Dynamical Systems ...

2D Flows ...

Limit cycle existence

(requires real work to show!)

Systems that can't have stable oscillations:

1. Simple harmonic oscillator
2. Gradient systems: $\dot{\vec{x}} = -\nabla V(\vec{x})$
3. Lyapunov systems

Example Dynamical Systems ...

2D Flows ...

Limit cycle existence
(requires real work to show!)

How to find limit cycles?

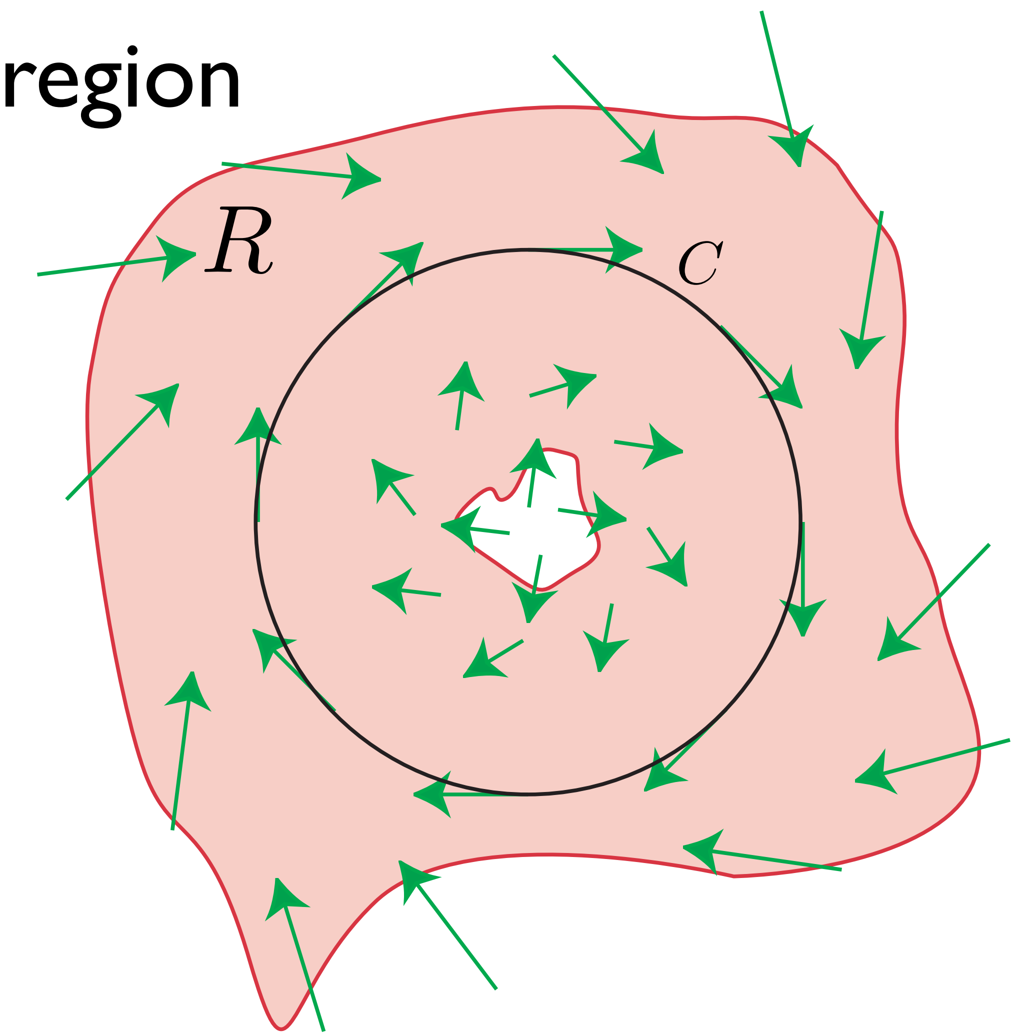
Poincaré-Bendixson Theorem:

(a) trajectory confined to trapping region

(b) no fixed points

then have limit cycle C

somewhere inside R.



Example Dynamical Systems ...

3D Flows:

Fixed points

Limit cycles

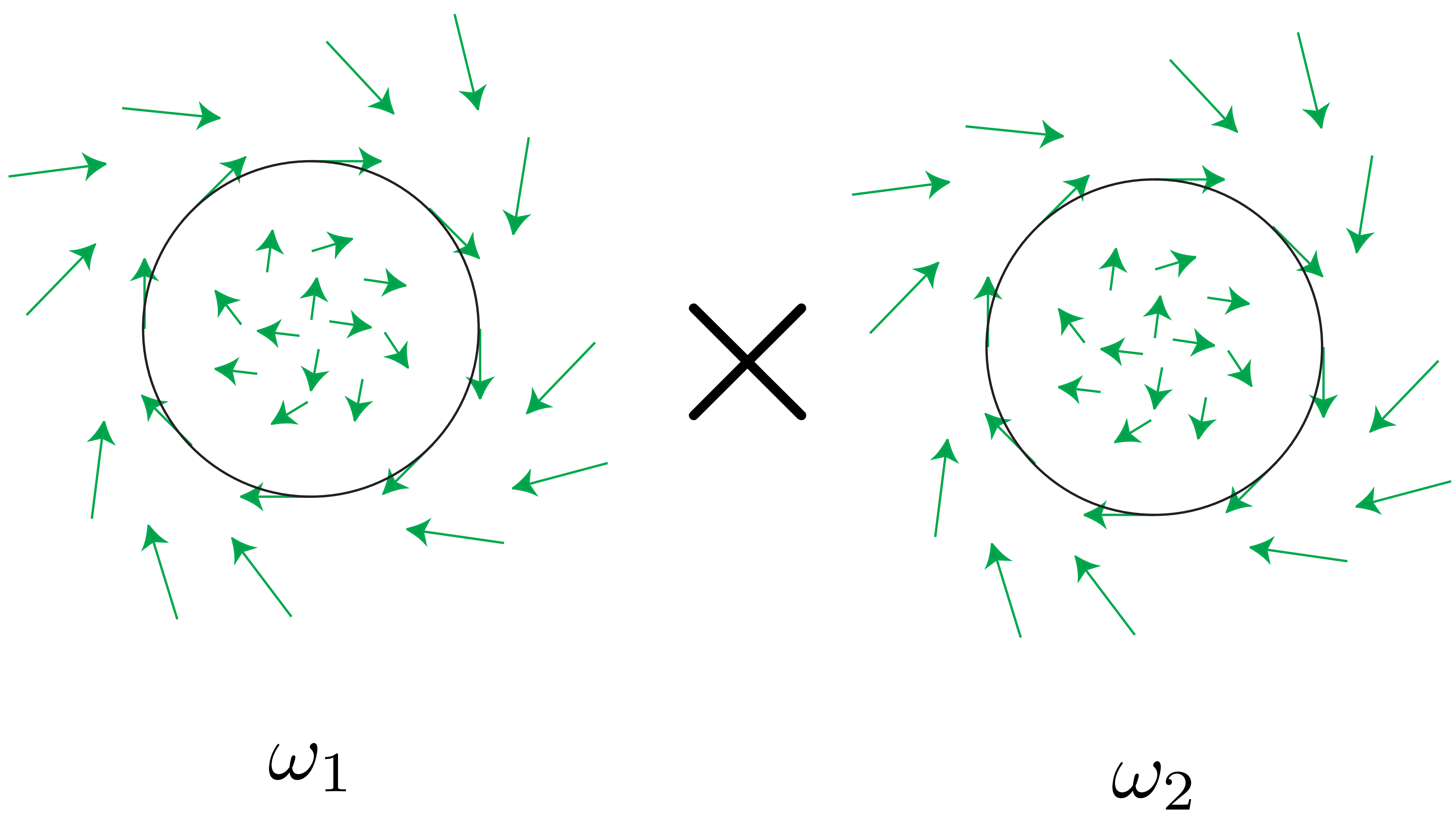
and ... ?

Example Dynamical Systems ...

3D Flows: Quasiperiodicity

product of two limit cycles:

two irrational frequencies $\omega_1 \neq \omega_2$

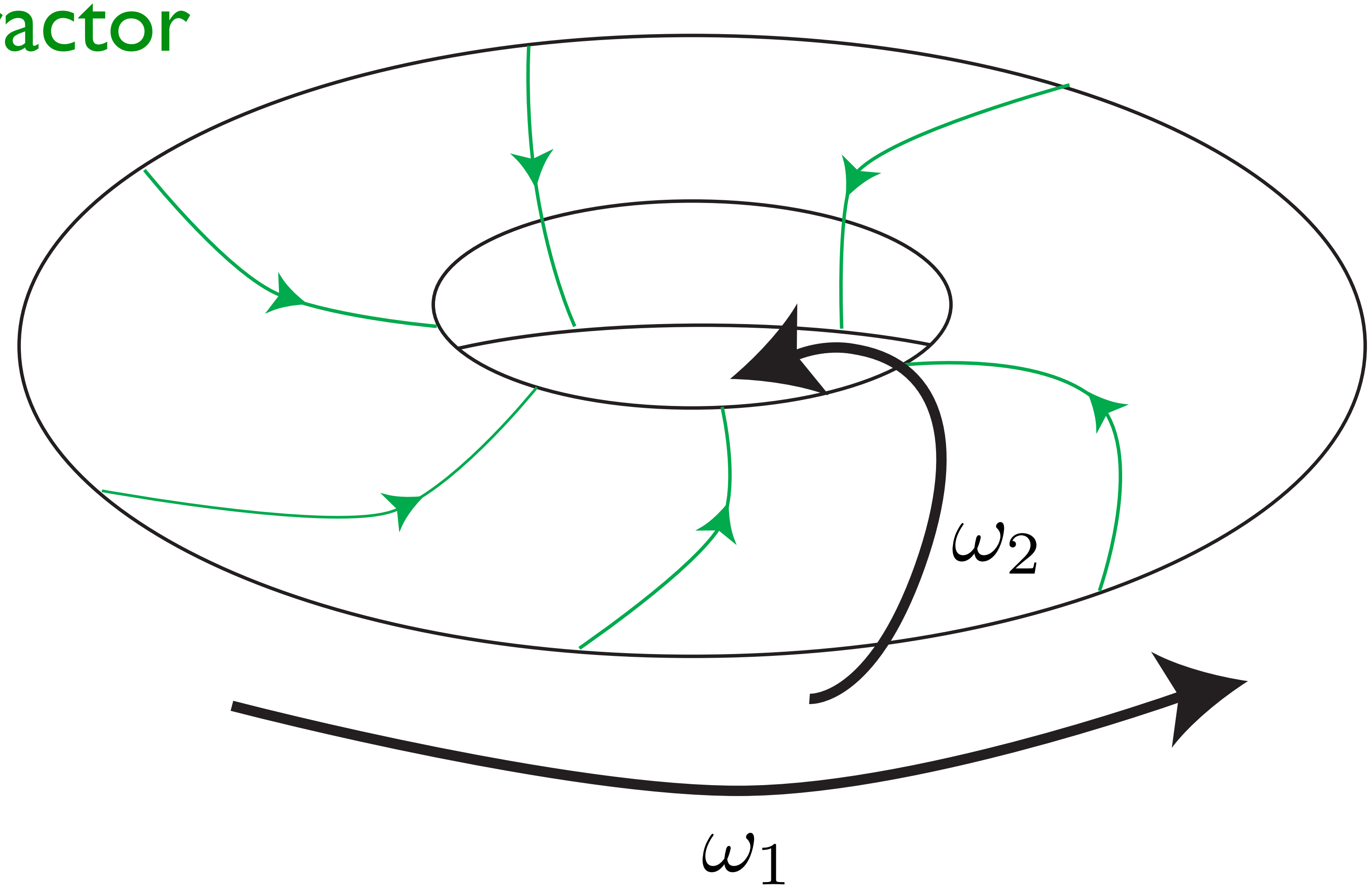


3D Flows: Quasiperiodicity

product of two limit cycles: two frequencies, $n \cdot w_1 \neq m \cdot w_2$

a new kind of behavior: *not possible* in 1D or 2D

Torus attractor



The Big, Big Picture (Bifurcations II) ...

Torus attractor in 3D Flow:

Driven van der Pol:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = A\sin(\omega t)$$

One way to verify 3D state space:

Write out equivalent first-order ODEs:

(Goal: RHS is time independent)

$$\begin{aligned}
\dot{x} &= y \\
\dot{y} &= \mu(1 - x^2)y - x + A\sin(\phi) \\
\dot{\phi} &= \omega
\end{aligned}$$

$$(x, y, \phi) \in \mathbb{R}^2 \times S^1$$

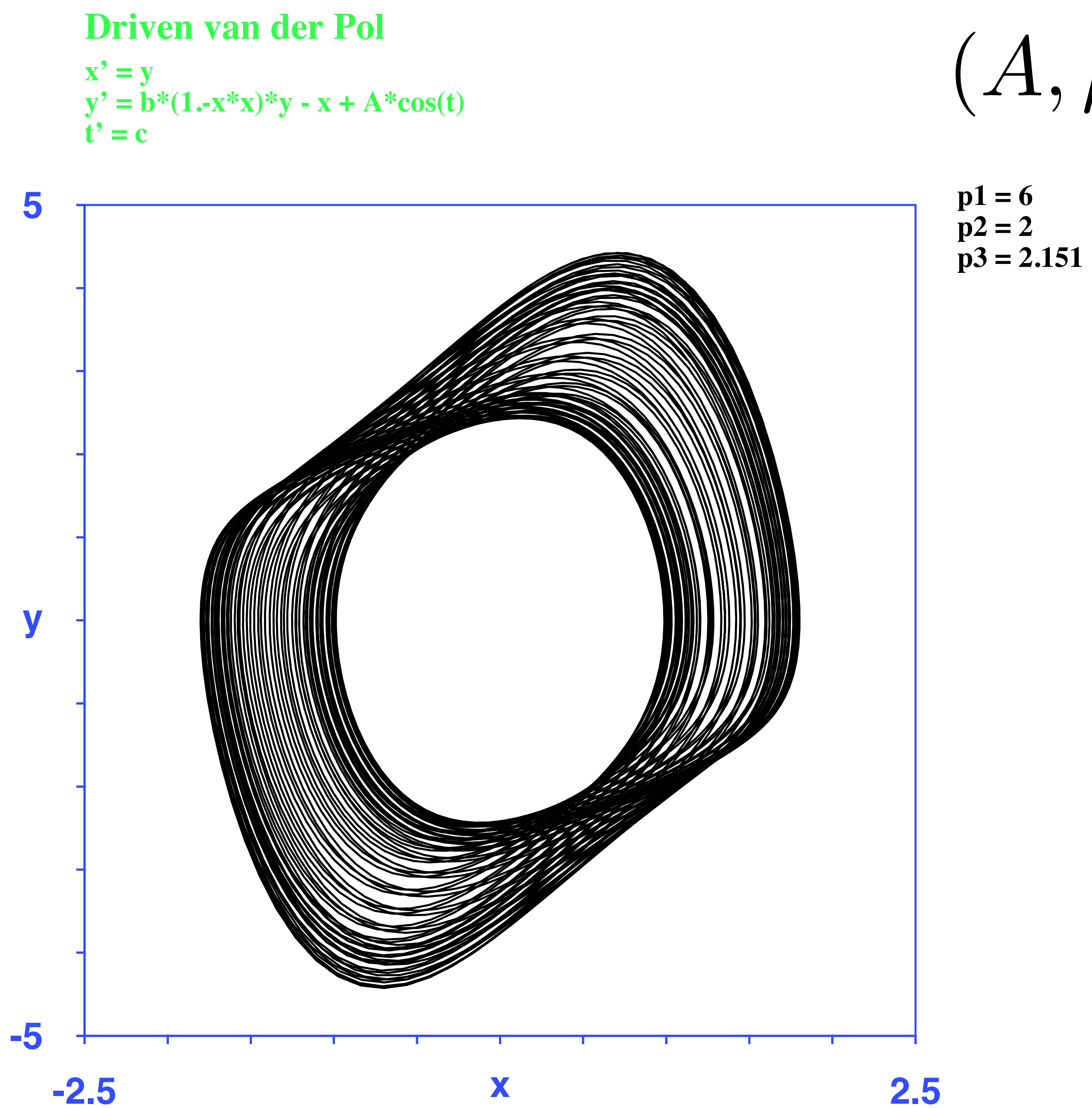
The Big, Big Picture (Bifurcations II) ...

Torus attractor in 3D Flow:

Driven van der Pol:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = A\sin(\omega t)$$

$$(A, \mu, \omega) = (6.0, 2.0, 2.151)$$



Example Dynamical Systems ...

3D Flows: Chaos

recurrent instability

one way to do this:
    Orbit reinjection near unstable fixed point

not possible in lower D flows

a new behavior type

Example Dynamical Systems ...

3D Flows: Chaos ...

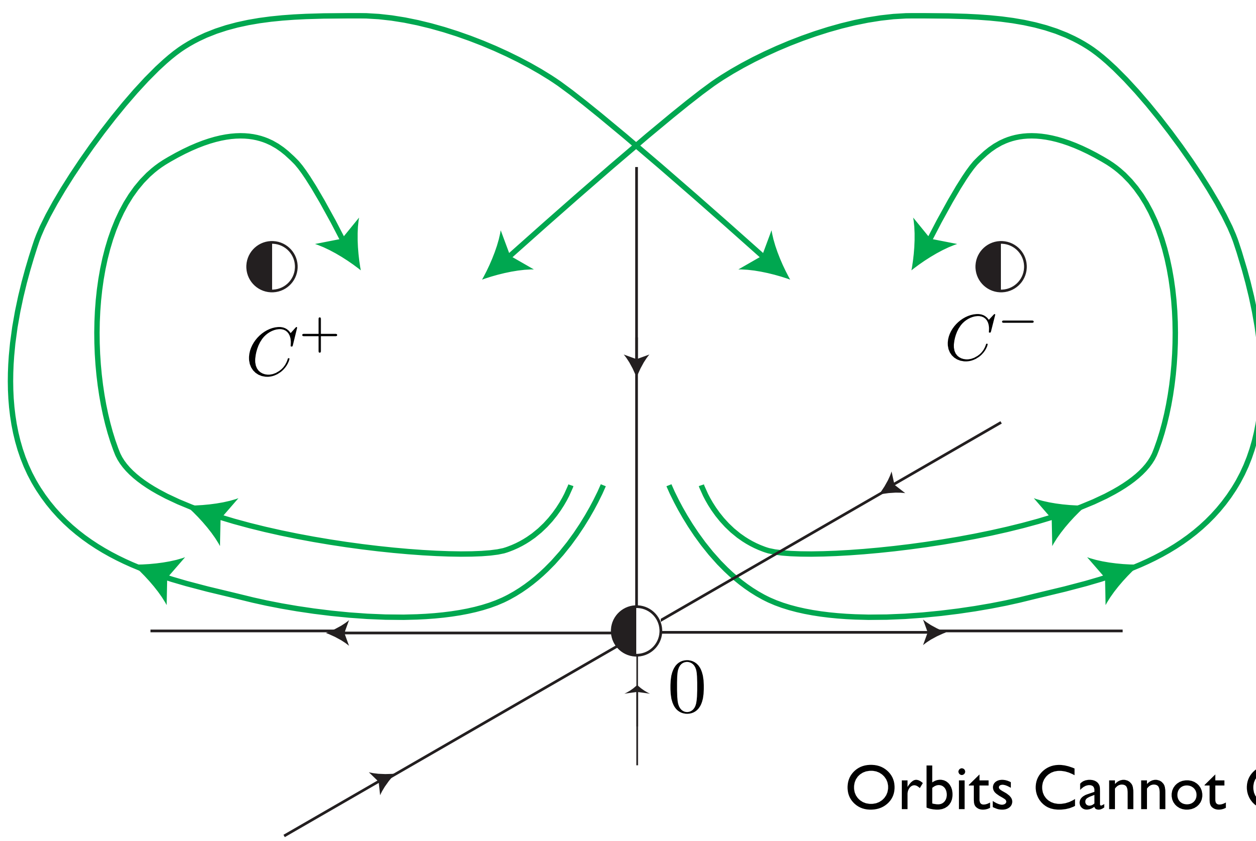
A topological construction:

saddle fixed point at origin: 0

1D unstable manifold: $\dim(W^u(\mathbf{0})) = 1$

2D stable manifold: $\dim(W^s(\mathbf{0})) = 2$

two fixed points: C^+ & C^-



Orbits Cannot Cross: Need 3D!

Does any ODE implement this flow design?

Example Dynamical Systems ...

3D Flows: Chaos ...

Does any ODE implement this design?

Yes, the Lorenz equations:

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= rx - y - xz \\
\dot{z} &= xy - bz
\end{aligned}
$$

Parameters: $\sigma, r, b > 0$

Exercise: Show fixed point at the origin can be a saddle, with 2 stable and 1 unstable directions

Exercise: Show there is a symmetry $(x, y) \rightarrow (-x, -y)$

3D Flows: Chaos ...

Lorenz ODE properties:

Trajectories stay in a bounded region near origin

No stable fixed points or stable limit cycles inside region

Volume shrinks to zero (everywhere inside):

$$\dot{V} = \int dV \; \nabla \cdot \vec{F}(\vec{x}) \qquad \nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$$

$$\nabla \cdot \vec{F}(\vec{x}) = \mathrm{Tr}(A) = -\sigma - 1 - b$$

$$\dot{V} = -(\sigma + 1 + b)V$$

$$V(t) = e^{-(\sigma+1+b)t}$$

Region volume shrinks exponentially fast!

$\mathrm{e.g.} : \sigma + 1 + b \approx 10$

What does the invariant set look like?

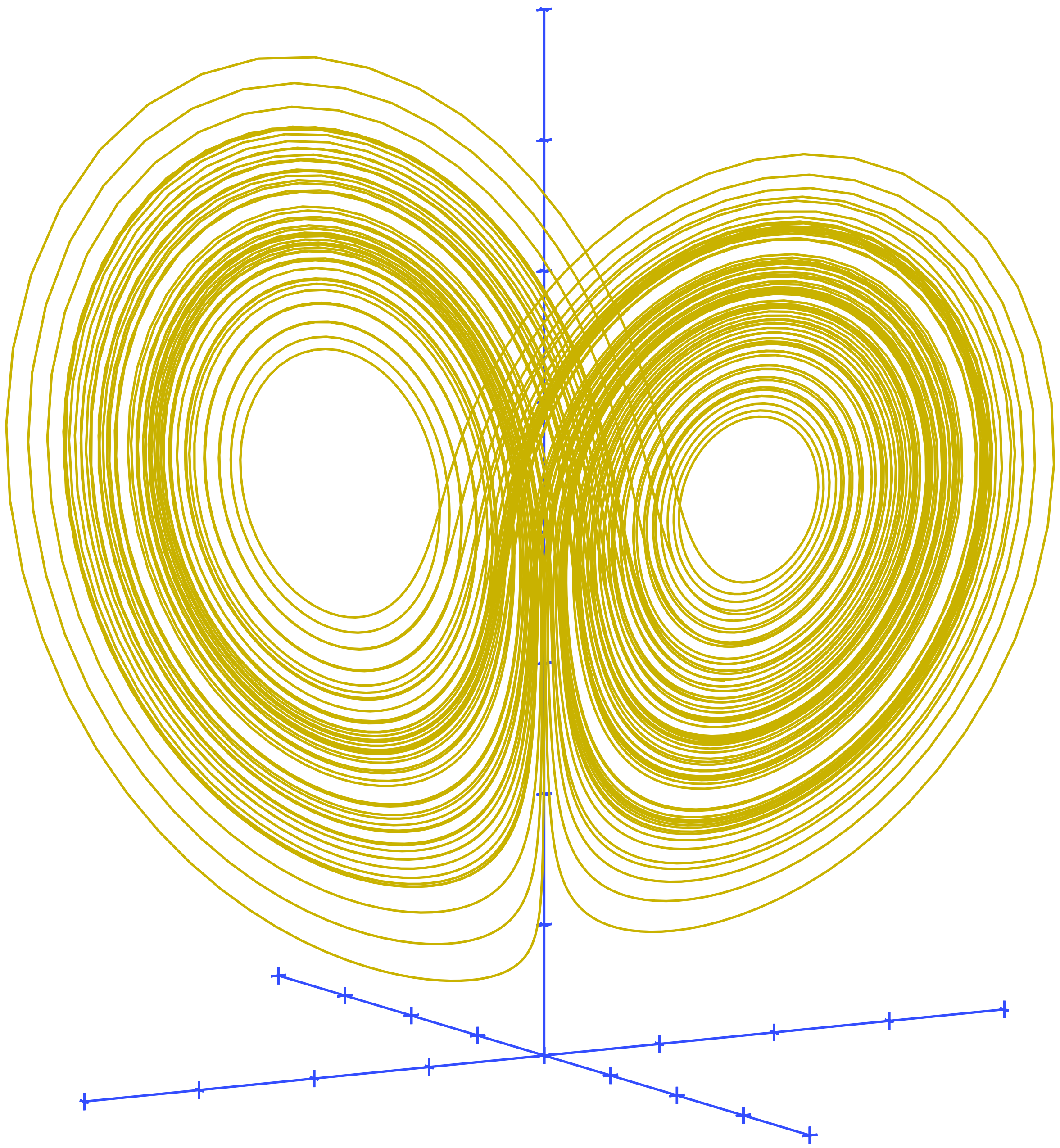
Example Dynamical Systems ...

3D Flows: Chaos ...

Lorenz simulation demo:

Chaotic attractor:

$$(\sigma, r, b) = (10, 28, 8/3)$$

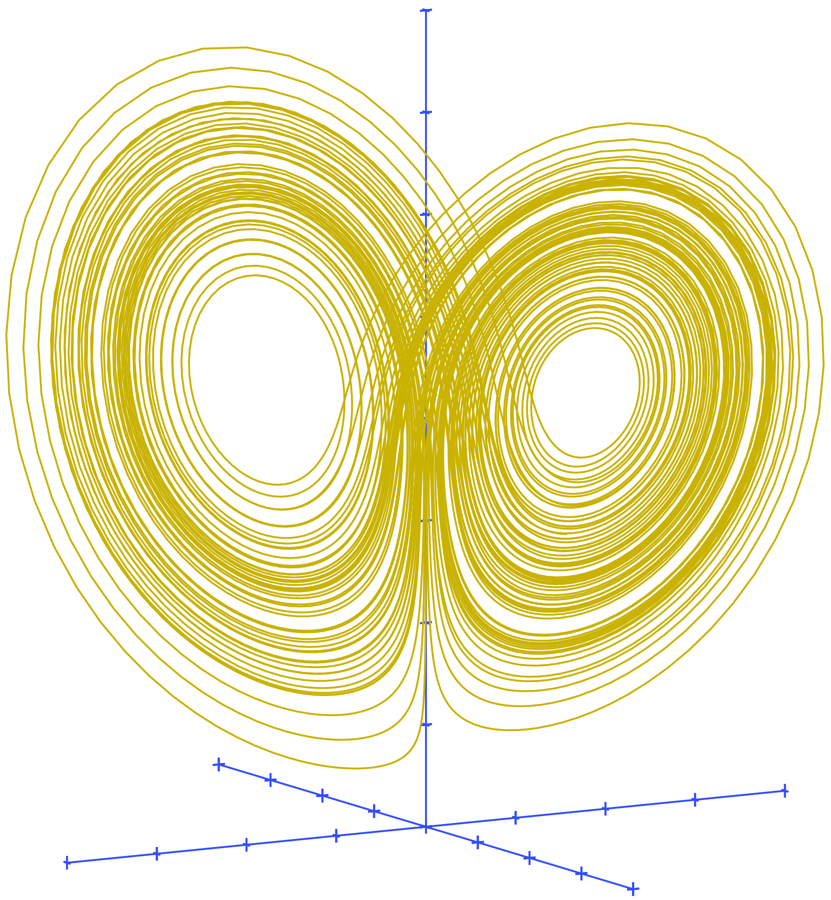


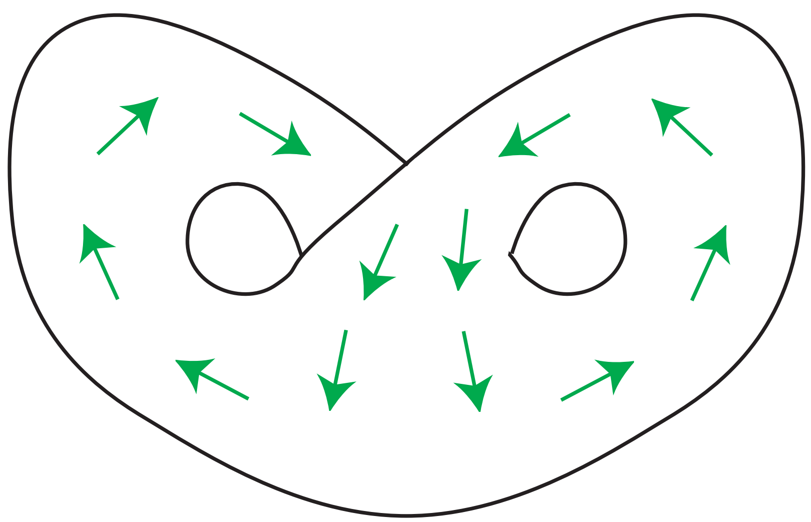
Demo:
- Time step = 0.001
- Parameters
- IC = (3,3,3)
- 3D view, orientation
- len = 100K

Example Dynamical Systems ...

3D Flows: Chaos ...

Lorenz attractor structure





3D Flows: Chaos ...

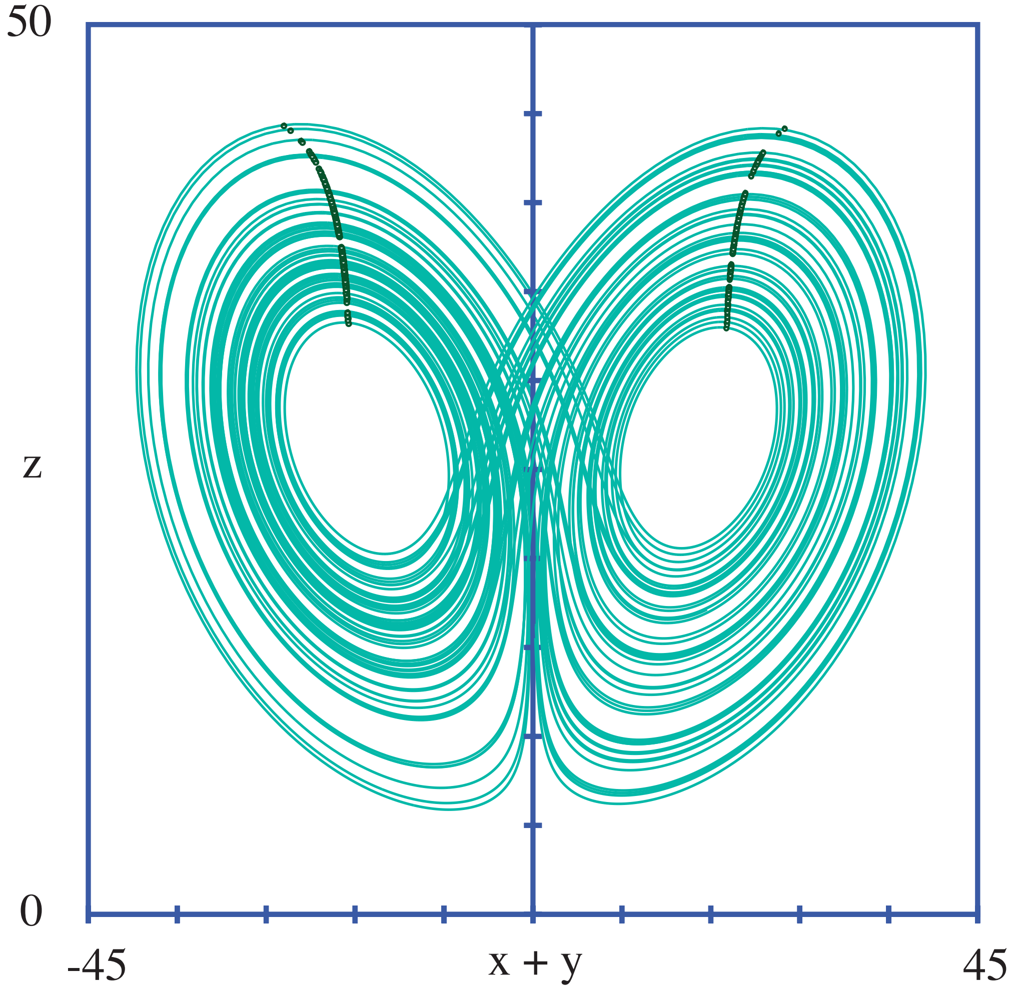
Lorenz simulation demo:

fixed point:

$$(\sigma, r, b) = (10, 15, 8/3)$$

Demo:
Time step = 0.0001
Parameters
IC = (3,3,3)
3D view, orientation
len = 100K

From Continuous-Time Flows to Discrete-Time Maps:



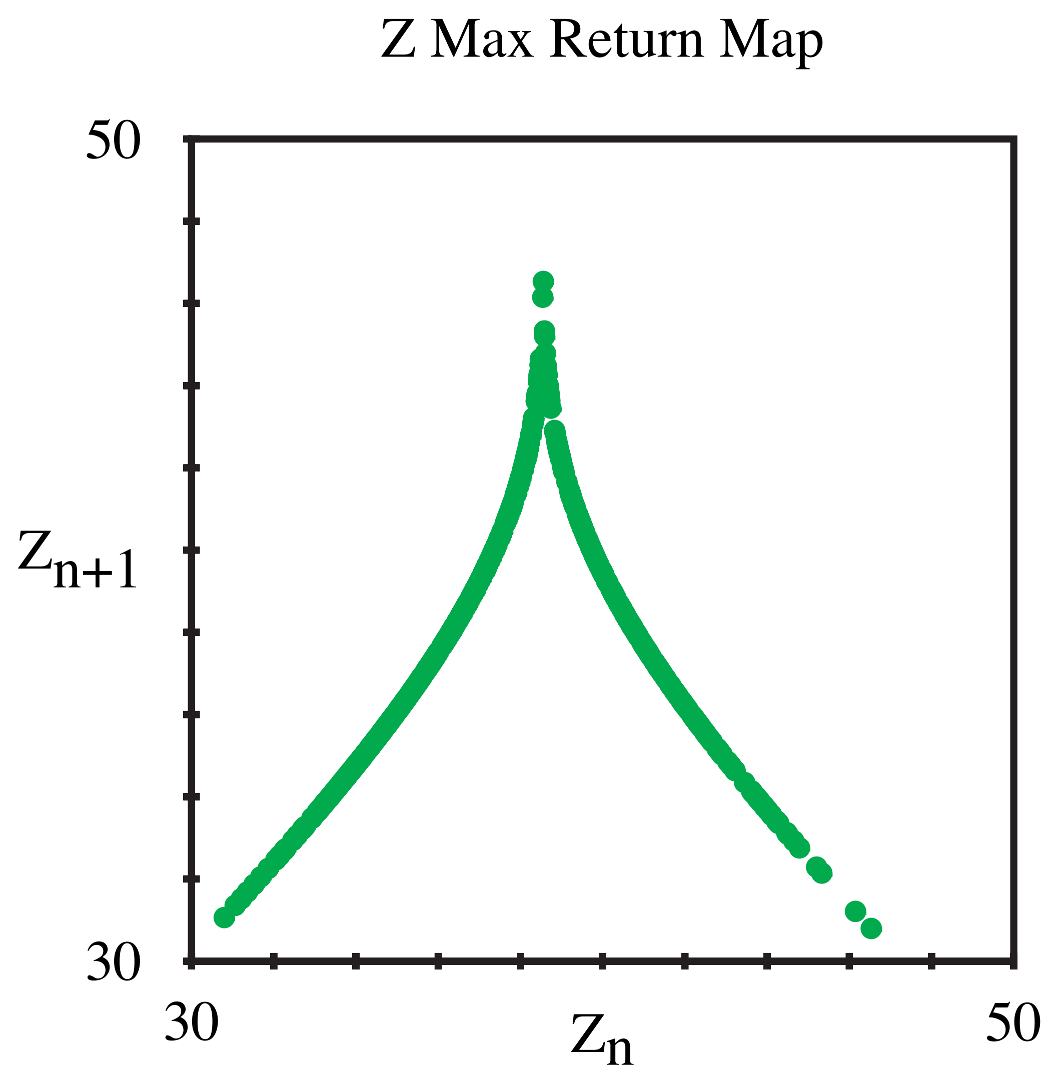
Series of z-maxima: $\widehat{z}_1, \widehat{z}_2, \widehat{z}_3, \ldots$

What happens if you plot
$\widehat{z}_{n+1}$ versus $\widehat{z}_n$?

Example Dynamical Systems ...

From Continuous-Time Flows to Discrete-Time Maps:

Max-z Return Map: $z_{n+1} = f(z_n)$

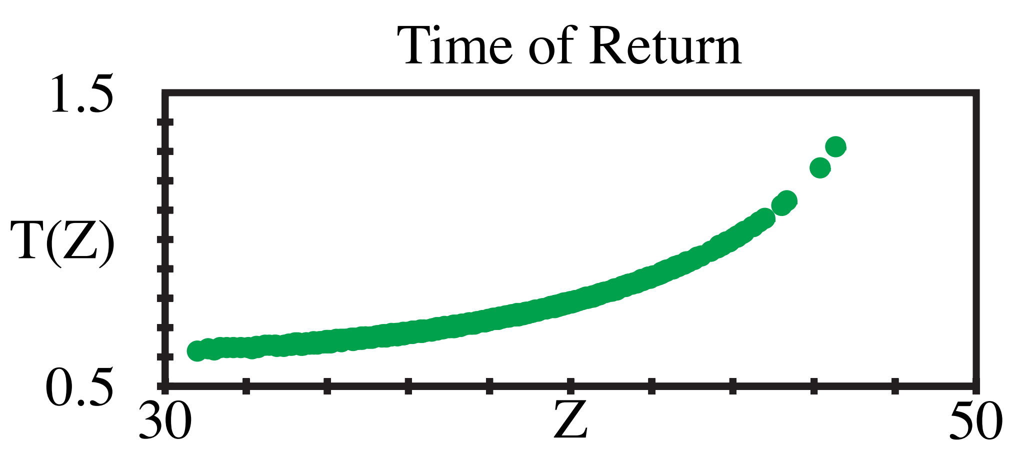


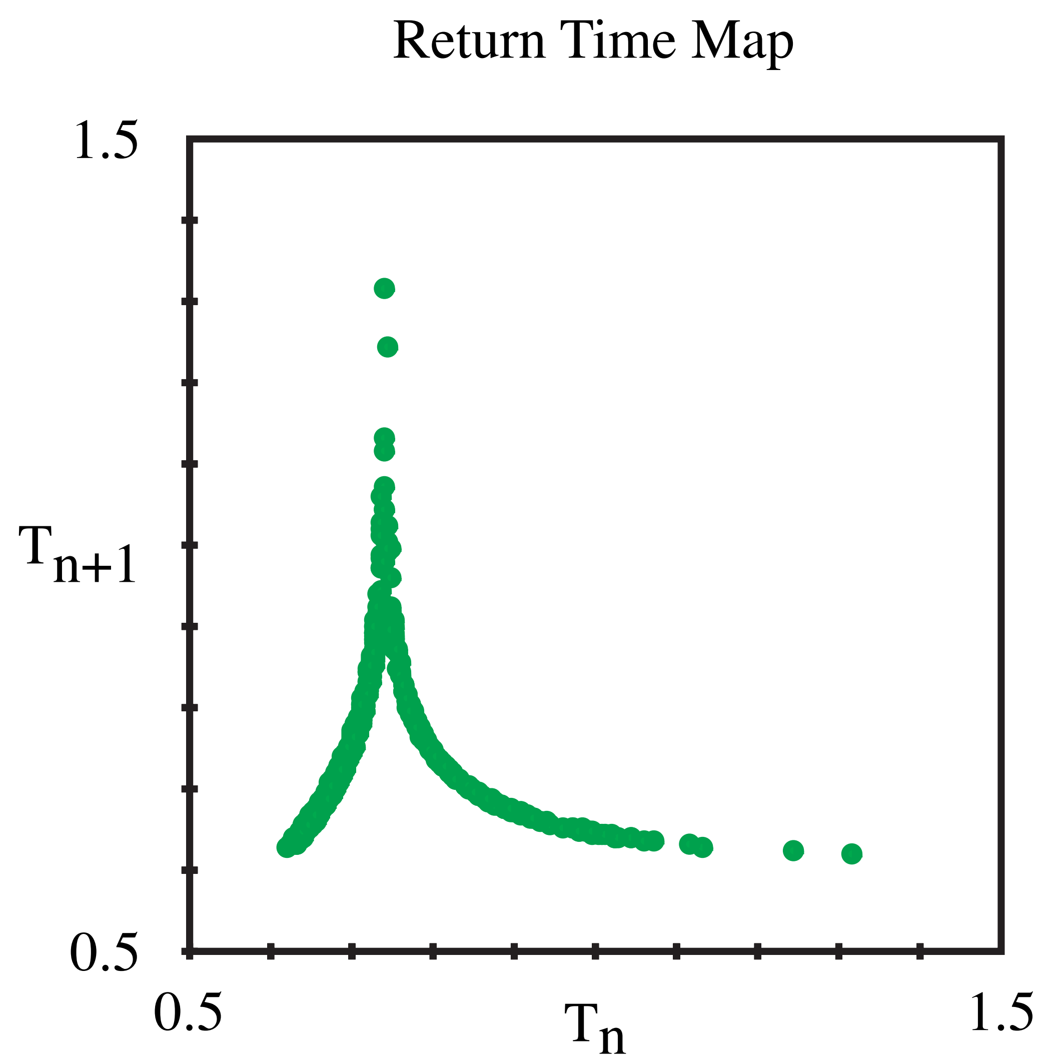
Example Dynamical Systems ...

From Continuous-Time Flows to Discrete-Time Maps:

Time of Return Function: $T(z_n)$

Return Time Map: $T_{n+1} = h(T_n)$





Lecture 3: Nonlinear Physics, Physics 150/250 (Spring 2010); Jim Crutchfield

Example Dynamical Systems ...

3D Flows ...

Lorenz reduces to a cusp 1D map:

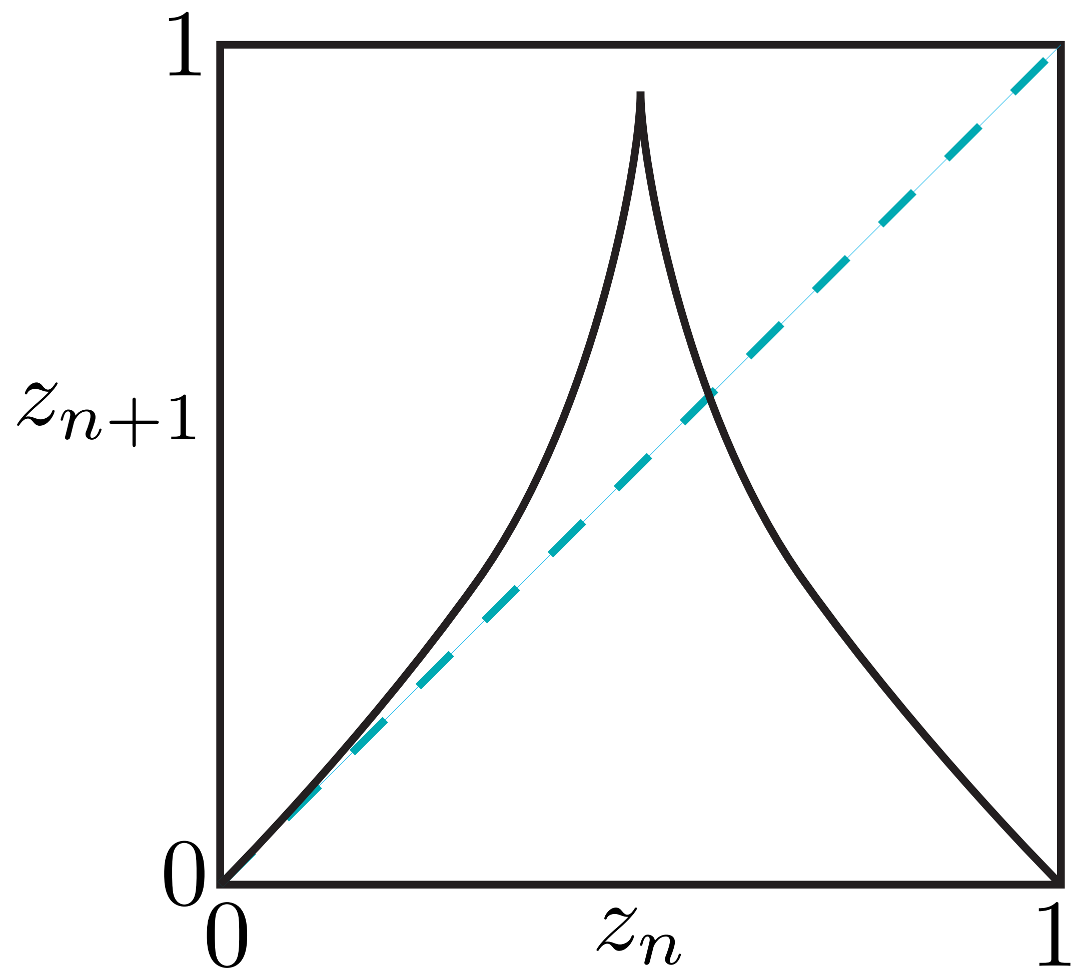
normalize to $z_n \in [0, 1]$

$$z_{n+1} = a(1 - |1 - 2z_n|^b)$$

Parameters:

height: $a > 0$

peak sharpness: $0 < b < 1$



3D Flows ...

Rössler equations

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c) \end{aligned}$$

Parameters: $a, b, c > 0$

Example Dynamical Systems ...

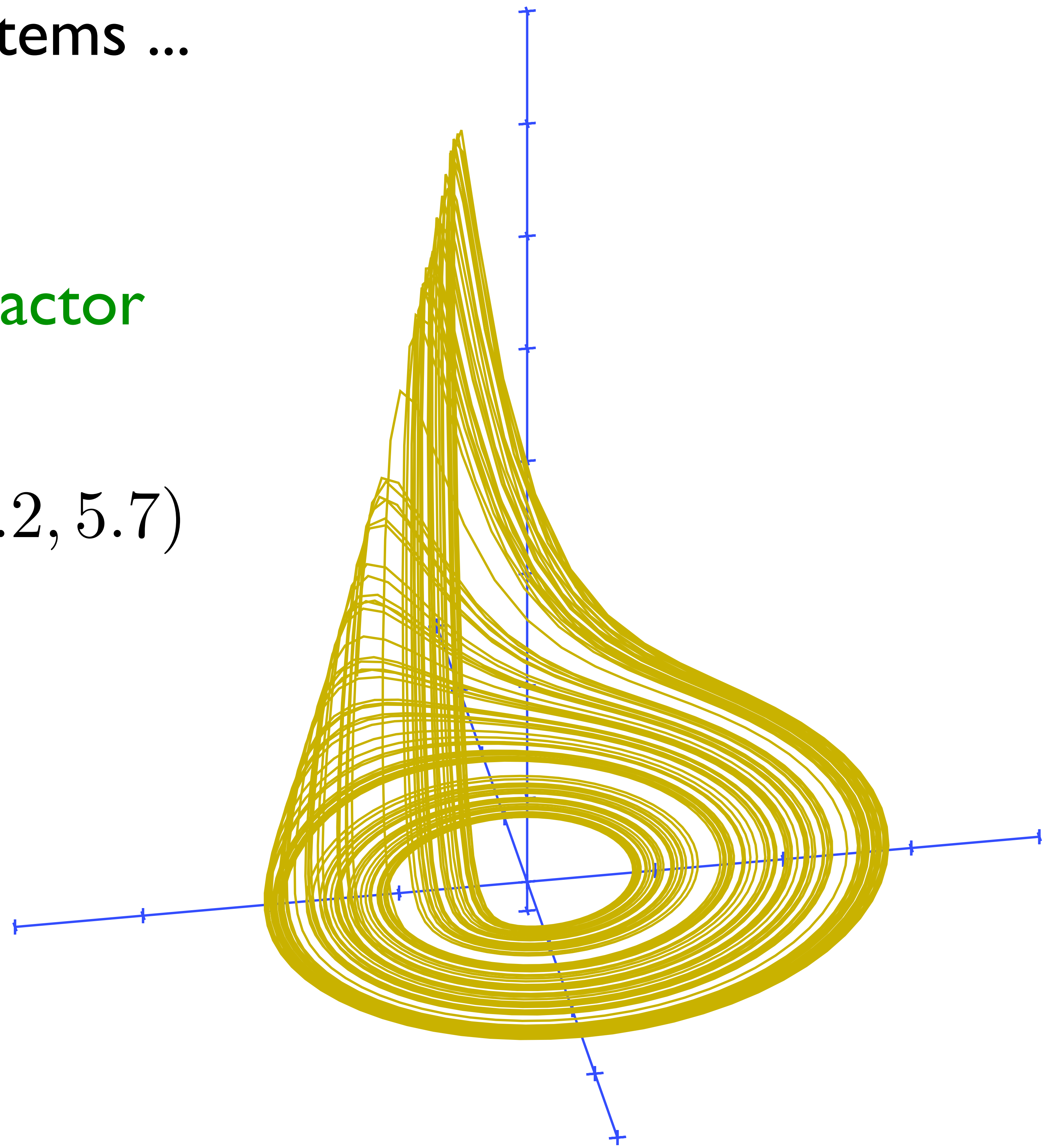
3D Flows ...

Rössler chaotic attractor

Parameters:

$$(a, b, c) = (0.2, 0.2, 5.7)$$

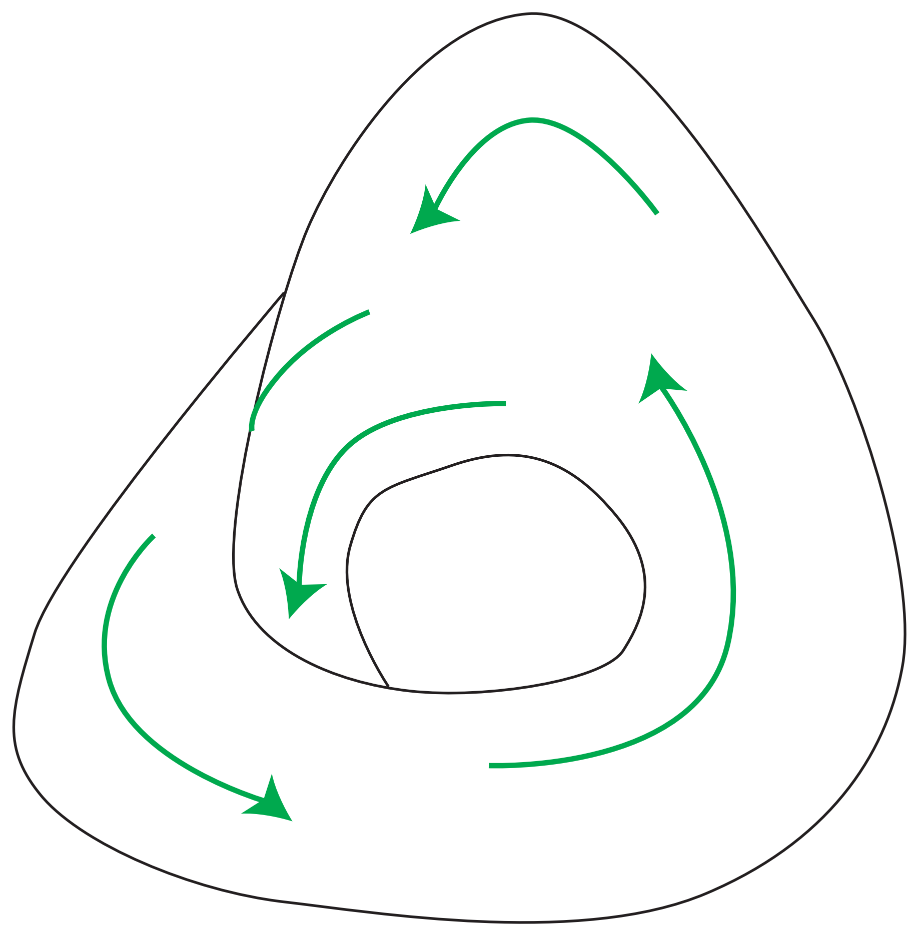
Demo:
- Time step = 0.01
- Parameters
- IC = (3,3,3)
- 3D view, orientation
- len = 100K



Example Dynamical Systems ...

3D Flows ...

Rössler branched manifold



Lecture 3: Nonlinear Physics, Physics 150/250 (Spring 2010); Jim Crutchfield

Example Dynamical Systems ...

3D Flows ...

Rössler limit cycle attractor

Parameters:

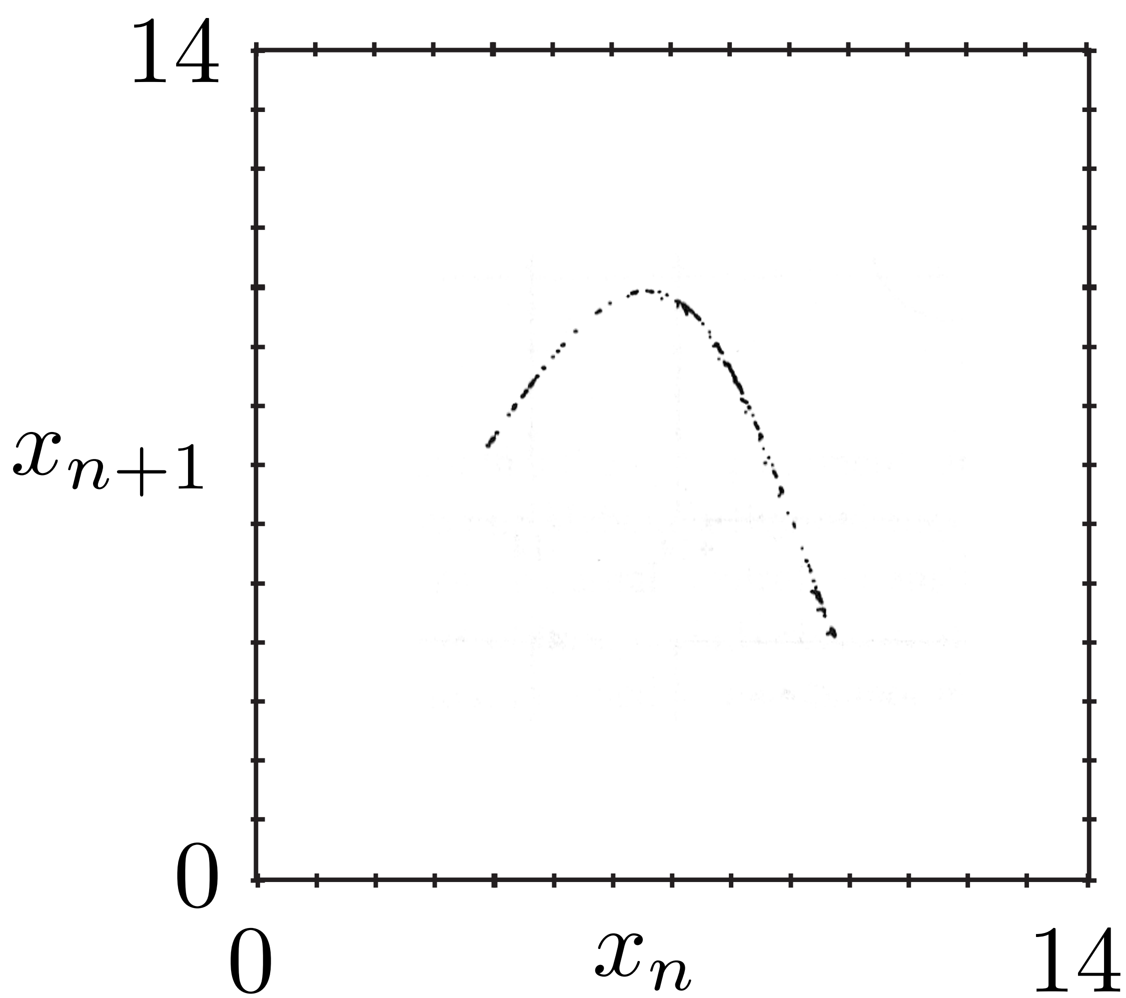
$$(a, b, c) = (0.2, 0.2, 2.0)$$

Demo:
- Time step = 0.01
- Parameters
- IC = (3,3,3)
- 3D view, orientation
- len = 100K

Example Dynamical Systems ...

3D Flows ...

Rössler maximum-x return map: $x_{n+1} = f(x_n)$



Example Dynamical Systems ...

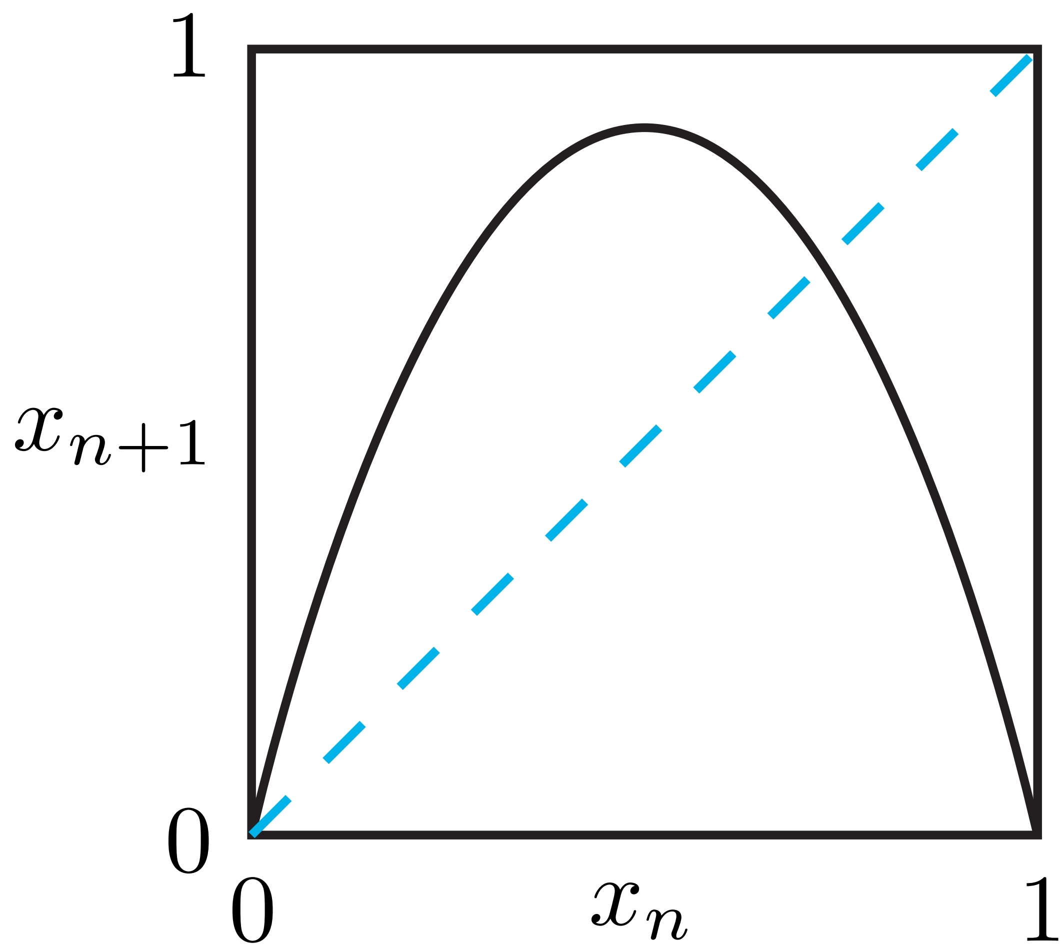
3D Flows ...

When normalized to $x_n \in [0, 1]$

get the Logistic Map:

$$x_{n+1} = r x_n (1 - x_n)$$

Parameter (height): $r \in [0, 4]$



Classification of Possible Behaviors

Dimension	Attractor
1	Fixed point
2	Fixed point, Limit cycle
3	Fixed Point, Limit Cycle, Torus, Chaotic
4	Above + Hyperchaos
5	?

Example Dynamical Systems ...

Lorenz:
$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= rx - y - xz \\ \dot{z} &= xy - bz\end{aligned} \qquad \sigma, r, b > 0$$

Rössler:
$$\begin{aligned}\dot{x} &= -y - z \\ \dot{y} &= x + ay \\ \dot{z} &= b + z(x - c)\end{aligned}$$

Cusp Map: $z_n \in [0, 1] \quad a > 0, 0 < b < 1$

$$z_{n+1} = a(1 - |1 - 2z_n|^b)$$

Logistic map:

$$x_{n+1} = rx_n(1 - x_n) \qquad x_n \in [0, 1] \qquad r \in [0, 4]$$

Play with these!

The Big Picture

Global view of the state space structures:
The attractor-basin portrait

Example Dynamical Systems ...

Reading for next lecture:

NDAC, Chapter 3.