The Big Picture

Python environment?

Discuss

- Examples of unpredictability

- *Chaos*, Scientific American (1986)

- *Odds*, Stanislaw Lem, The New Yorker (1974)

Email homework to me:
chaos@cse.ucdavis.edu

Nonlinear Physics:
Modeling Chaos and Complexity

Jim Crutchfield
chaos@cse.ucdavis.edu; http://cse.ucdavis.edu/~chaos

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WWW: http://cse.ucdavis.edu/~chaos/courses/nlp/

Questionnaire

1. Name: ____________________
2. Graduate or Undergraduate (circle one) Auditors
   2 7 2
3. Email address: ____________________
4. Major/Field: ____________________
5. What programming language(s) have you used?
   (circle all appropriate)
   C or C++ or Java or Fortran or Python or Perl or Other ________
   7 9 4 1 5 2
6. What level of programming experience do you have?
   (circle one)
   Little or Moderate or Extensive
   1 5 3
7. Are you familiar with Unix? Yes or No (circle one)
   8 1
8. Do you have a laptop? Yes or No (circle one)
   9 0
9. Which OS(es) does it run?
   (circle all appropriate)
   Windows or OS X or Linux
   3 5 3
10. Do you have a desktop machine? Yes or No (circle one)
11. Which OS(es) does it run?
    (circle all appropriate)
    Windows or OS X or Linux
    1 1 2

PHY	3
MAT	5
CSE	1

JavaScript	1
Lisp	1
Ruby	1
Math'a	1

The Big Picture ...

The Pendulum

The Big Picture ...

Qualitative Dynamics (Reading: *NDAC*, Chapters 1 and 2)

What is it?

Analyze nonlinear systems *without* solving the equations.

Why is it needed?

In general, nonlinear systems cannot be solved in closed form.

Three tools:

- Statistics
- Computation: e.g., simulation
- Mathematics: Dynamical Systems Theory

Why each is good.

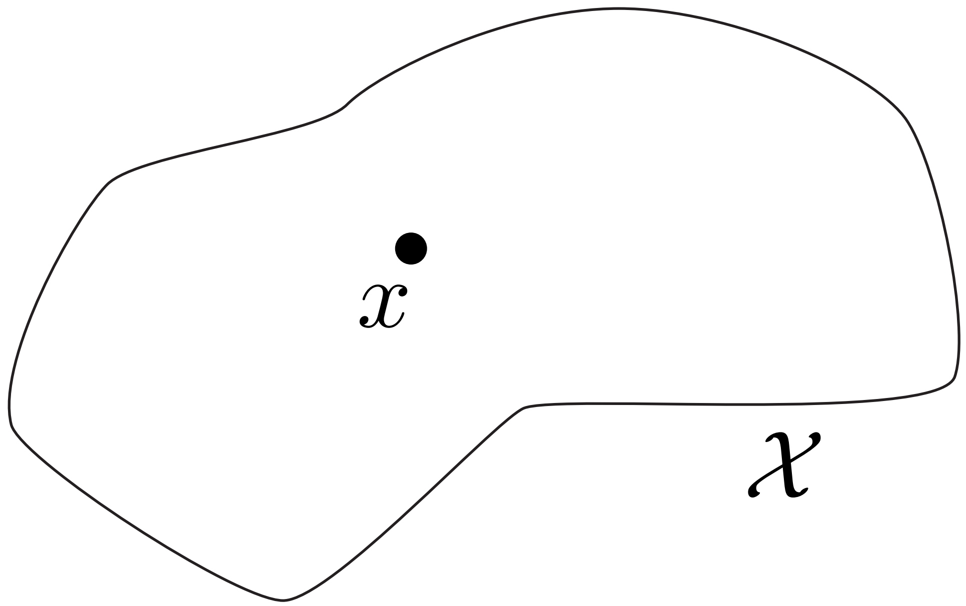
Why each fails in some way.

The Big Picture ...

Dynamical System: $\{\mathcal{X}, \mathcal{T}\}$

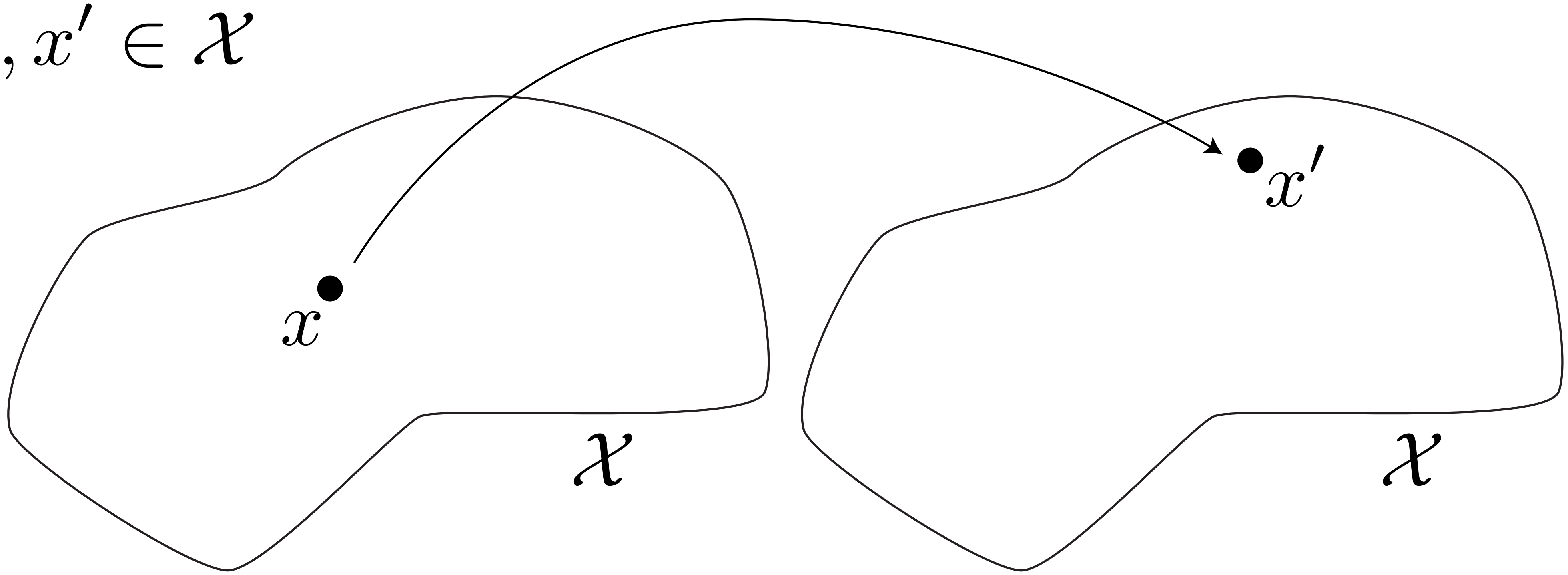
State Space: $\mathcal{X}$

State: $x \in \mathcal{X}$



Dynamic: $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$

$x, x' \in \mathcal{X}$

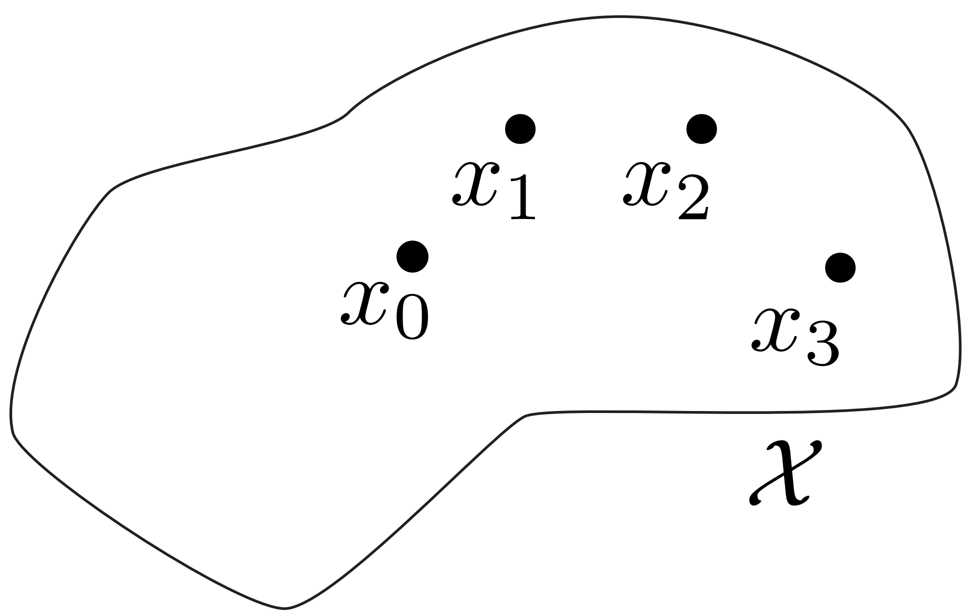


The Big Picture ...

Dynamical System ... $\{\mathcal{X}, \mathcal{T}\}$

Initial Condition: $x_0 \in \mathcal{X}$

Behavior: $x_0, x_1, x_2, x_2, \ldots$



The Big Picture ...

Dynamical System ...

For example, discrete time ...

Map: $\vec{x}_{t+1} = \vec{F}(\vec{x}_t) \qquad t = 0, 1, 2, \ldots$

State: $\vec{x}_t \in \mathbf{R}^n \qquad \vec{x} = (x_1, x_2, \ldots, x_n)$

State Space

Dimension: n

Initial condition: $\vec{x}_0$

Dynamic: $\vec{F} : \mathbf{R}^n \rightarrow \mathbf{R}^n \qquad \vec{F} = (f_1, f_2, \ldots, f_n)$

"Solution": $\vec{x}_0, \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \ldots$

The Big Picture ...

Dynamical System ...

For example, continuous time ...

Ordinary differential equation (ODE): $\dot{\vec{x}} = \vec{F}(\vec{x})$ $\qquad \left(\dot{\square} = \frac{d}{dt} \right)$

State: $\vec{x}(t) \in \mathbf{R}^n$ $\qquad \vec{x} = (x_1, x_2, \ldots, x_n)$

Dimension: n

Initial condition: $\vec{x}(0)$

Dynamic: $\vec{F} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ $\qquad \vec{F} = (f_1, f_2, \ldots, f_n)$

The Big Picture ...

Flow field for an ODE (aka Phase Portrait)

Geometric view of an ODE:

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$$

$$\frac{d\vec{x}}{dt} \approx \frac{\Delta\vec{x}}{\Delta t} = \frac{\vec{x}' - \vec{x}}{\Delta t}$$

$$\vec{x}' = \vec{x} + \Delta t \cdot \vec{F}(\vec{x})$$

Each state $\vec{x}$ has a vector attached $\vec{F}(\vec{x})$

that says to what next state to go: $\vec{x}' = \vec{x} + \Delta t \cdot \vec{F}(\vec{x})$.

The Big Picture ...

Flow field for an ODE (aka Phase Portrait)

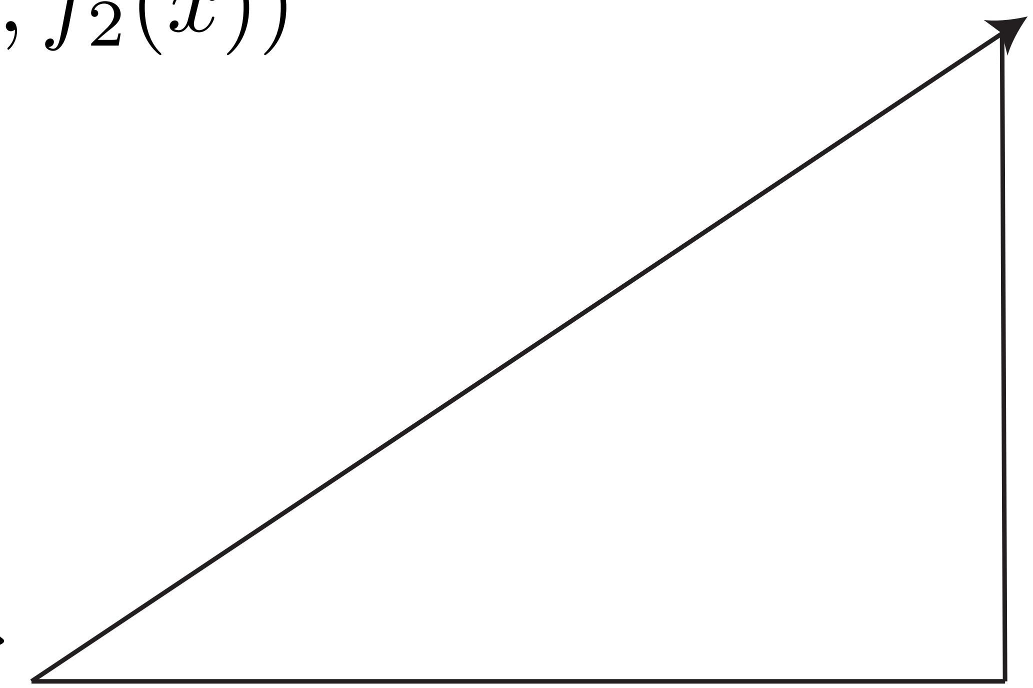
Geometric view of an ODE ...

$$\mathcal{X} = \mathbf{R}^2$$

$$\vec{x} = (x_1, x_2)$$

$$\vec{F} = (f_1(\vec{x}), f_2(\vec{x}))$$

$$\vec{x}' = \vec{x} + \Delta t \vec{F}(\vec{x})$$



$$\vec{x}$$

$$\Delta x_2 = \Delta t f_2(\vec{x})$$

$$\Delta x_1 = \Delta t f_1(\vec{x})$$

The Big Picture ...

Geometric view of an ODE ...

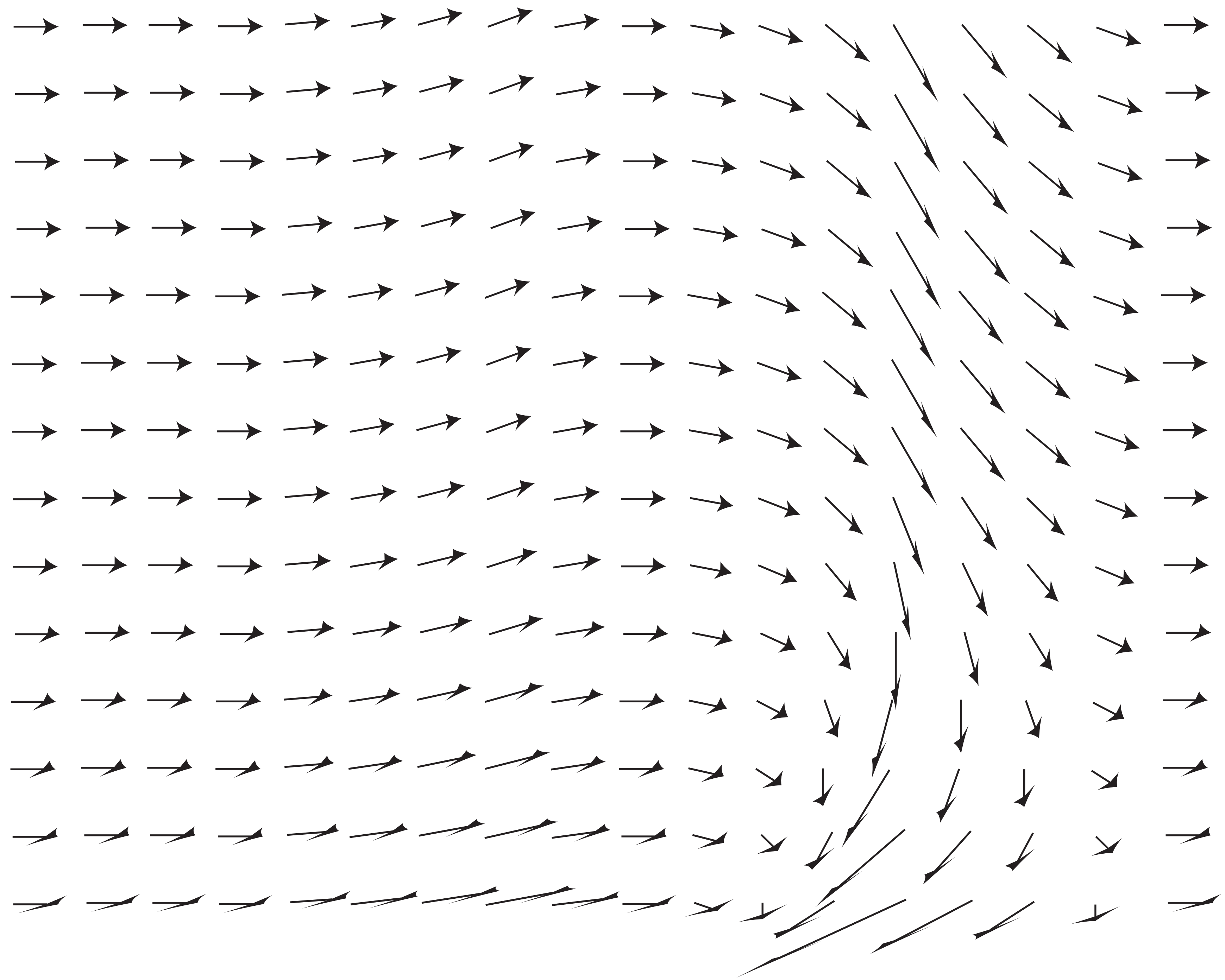
Vector field (aka Phase Portrait):

$\mathcal{X} = \mathbf{R}^2$

A set of rules:

Each state has a vector attached

That says to what next state to go



Lecture 2: Nonlinear Physics, Physics 150/250 (Spring 2010); Jim Crutchfield

The Big Picture ...

Solving the ODE: *Integrate* the *differential* equation! $\dot{\vec{x}} = \vec{F}(\vec{x})$

$$\vec{x}(T) = \vec{x}(0) + \int_0^T dt \; \dot{\vec{x}}(t)$$

$$\vec{x}(T) = \vec{x}(0) + \int_0^T dt \; \vec{F}(\vec{x}(t))$$

Time-T Flow: ϕ_T

The solution of the ODE, starting from a given IC

$$\vec{x}(T) = \phi_T(\vec{x}(0))$$

$$\phi_T : \mathcal{X} \to \mathcal{X}$$

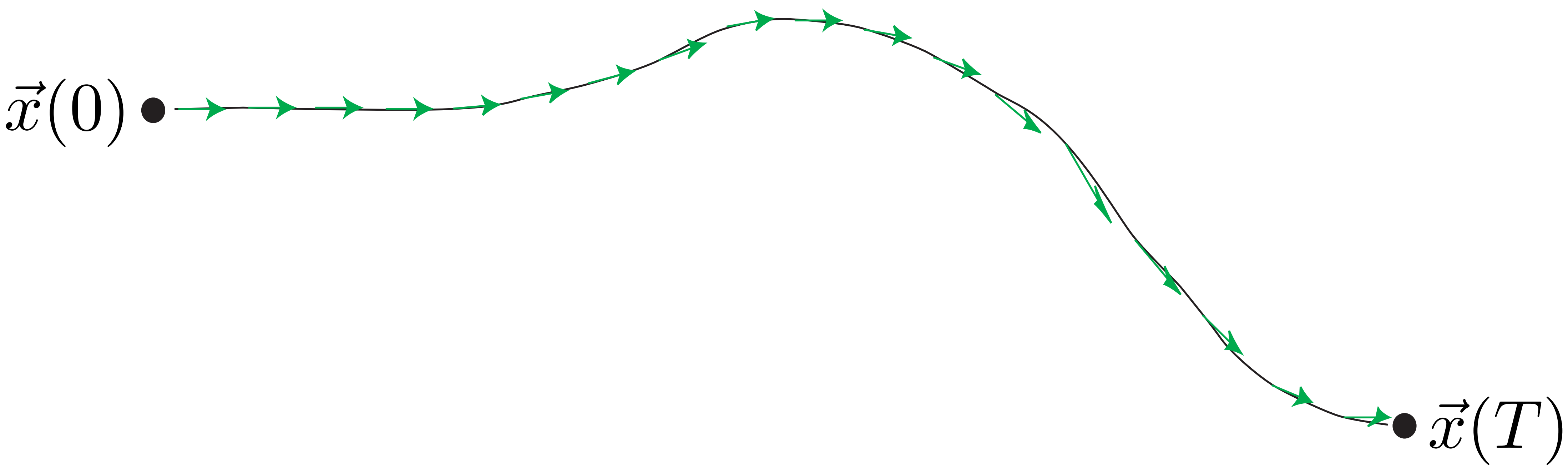
The Big Picture ...

Trajectory or Orbit:

the solution,

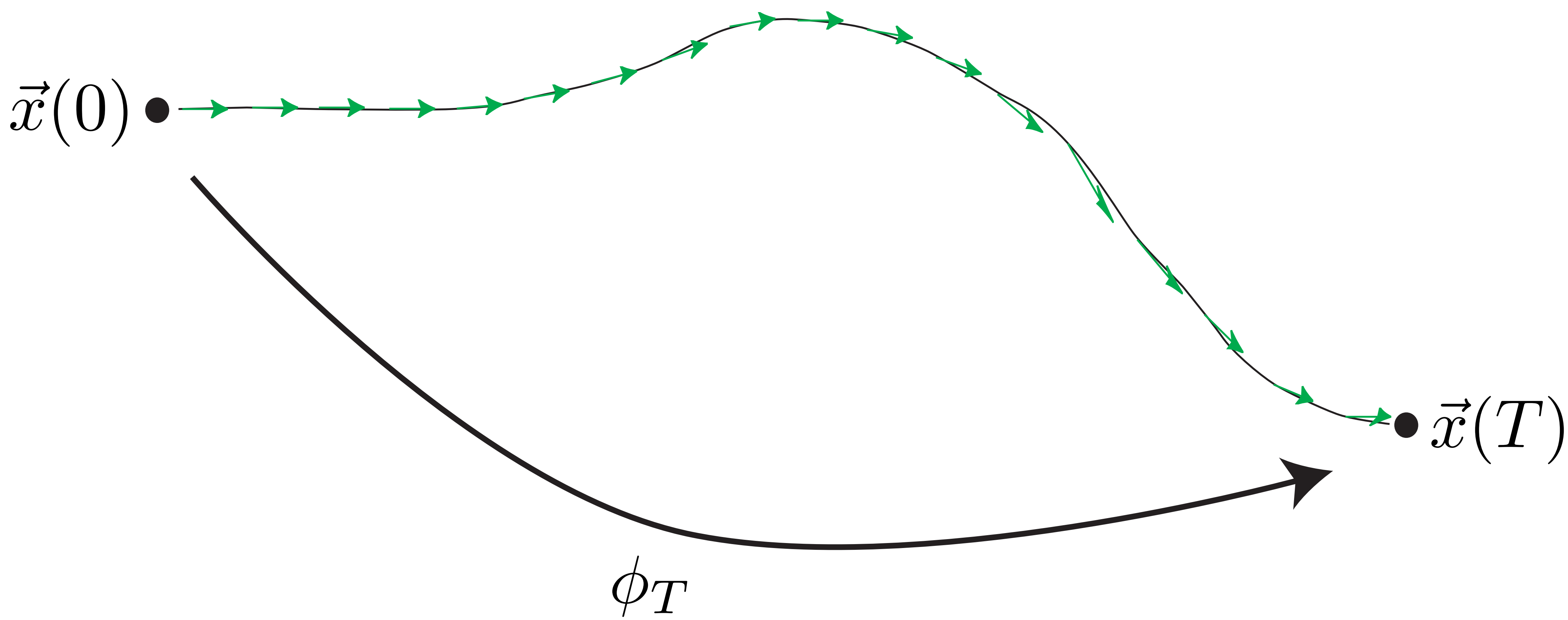
starting from some IC

simply follow the arrows



The Big Picture ...

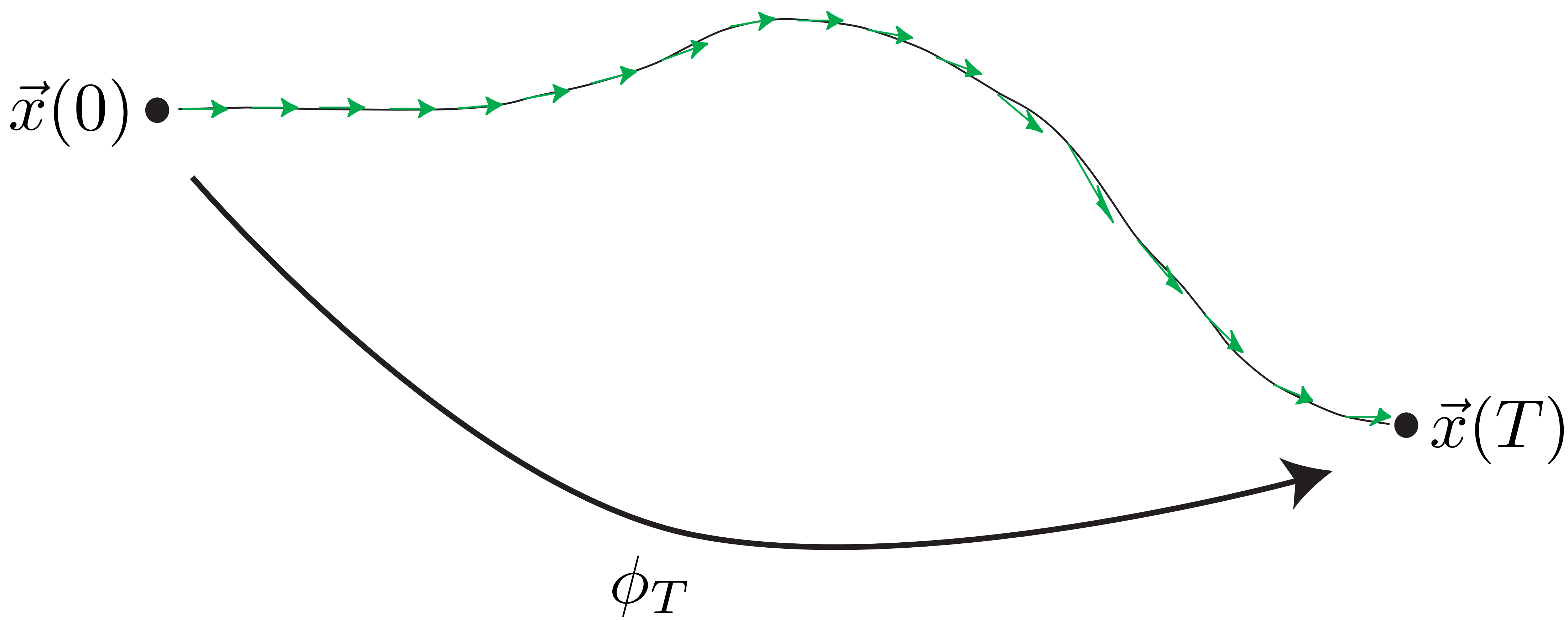
Time-T Flow: $\vec{x}(T) = \phi_T(\vec{x}(0))$



Point: ODE is only instantaneous,
Time-T Flow gives state for *any* time t

The Big Picture ...

Time-T Flow: $\vec{x}(T) = \phi_T(\vec{x}(0))$



Point: ODE is only instantaneous,
behavior is the *integrated*, *long-term* result.

The Big Picture ...

Example:

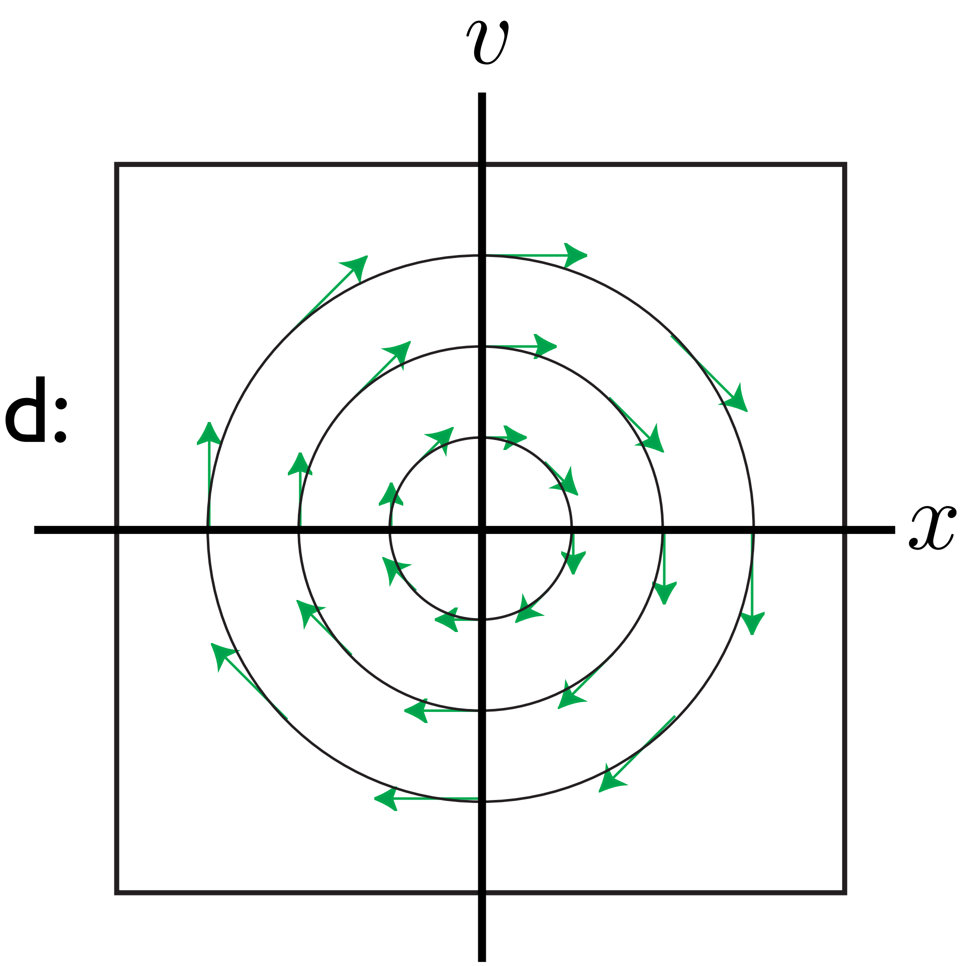
Simple Harmonic Oscillator: $\ddot{x} = -x$

As two coupled, first-order DEs:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -x \end{aligned}$$

with initial condition: (x_0, v_0)

Vector field:



Time-T flow (aka The Solution):

$$\phi_T(x_0, v_0) = (A\cos(T + \omega_0), A\sin(T + \omega_0))$$

$$A = \sqrt{x_0^2 + v_0^2} \qquad \omega_0 = \tan^{-1}\frac{v_0}{x_0}$$

The Big Picture ...

Invariant set: $\Lambda \subset \mathcal{X}$

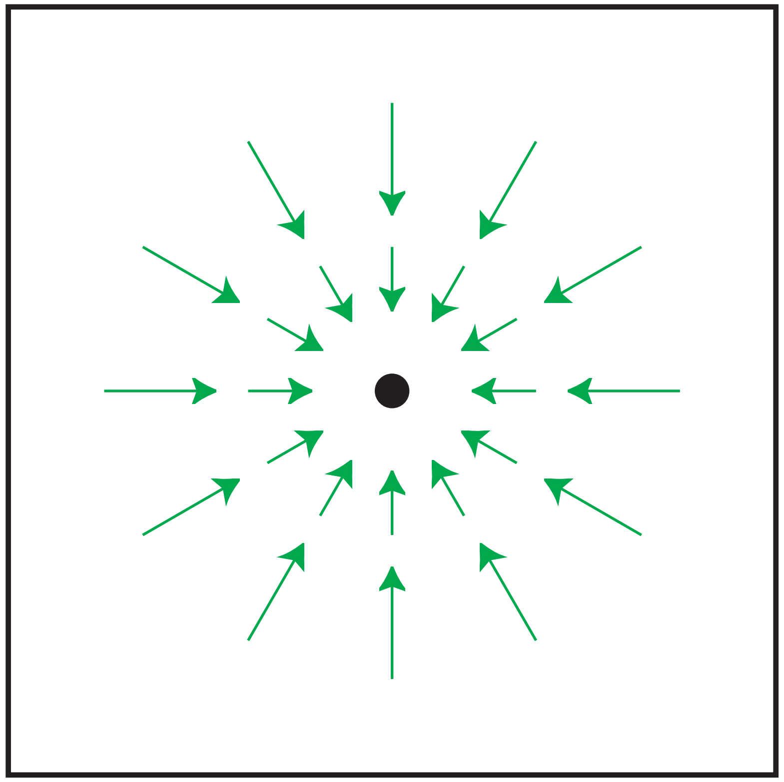
Set mapped into itself by the flow: $\Lambda = \phi_T(\Lambda)$

The Big Picture ...

Invariant set: $\Lambda \subset \mathcal{X}$

Set mapped into itself by the flow: $\Lambda = \phi_T(\Lambda)$

Example: Invariant point



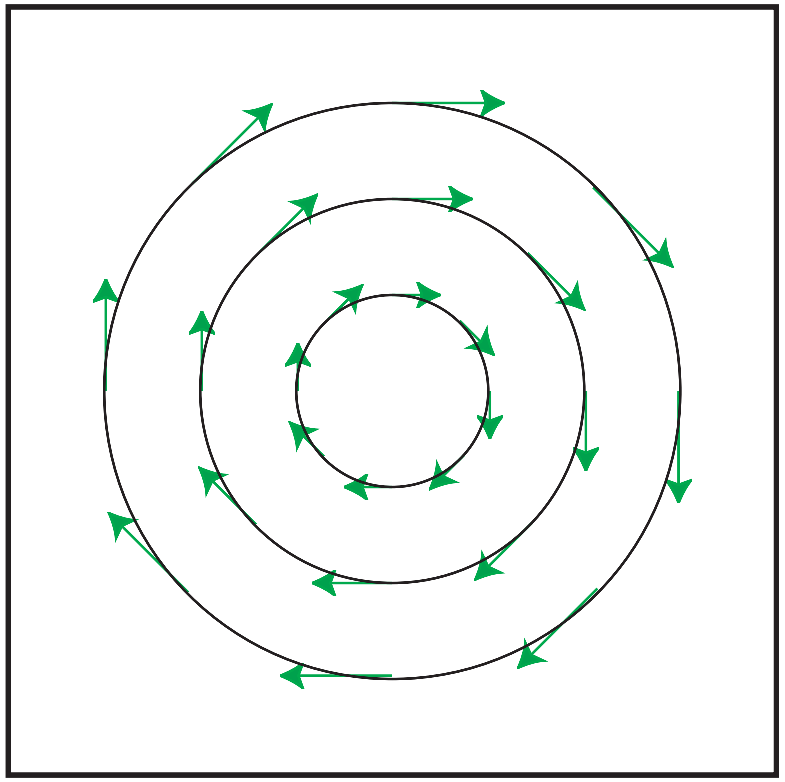
Fixed Point

The Big Picture ...

Invariant set: $\Lambda \subset \mathcal{X}$

Set mapped into itself by the flow: $\Lambda = \phi_T(\Lambda)$

For example: Invariant circles



Λ:

Any circle

Entire plane

Pure Rotation

(Simple Harmonic Oscillator)

The Big Picture ...

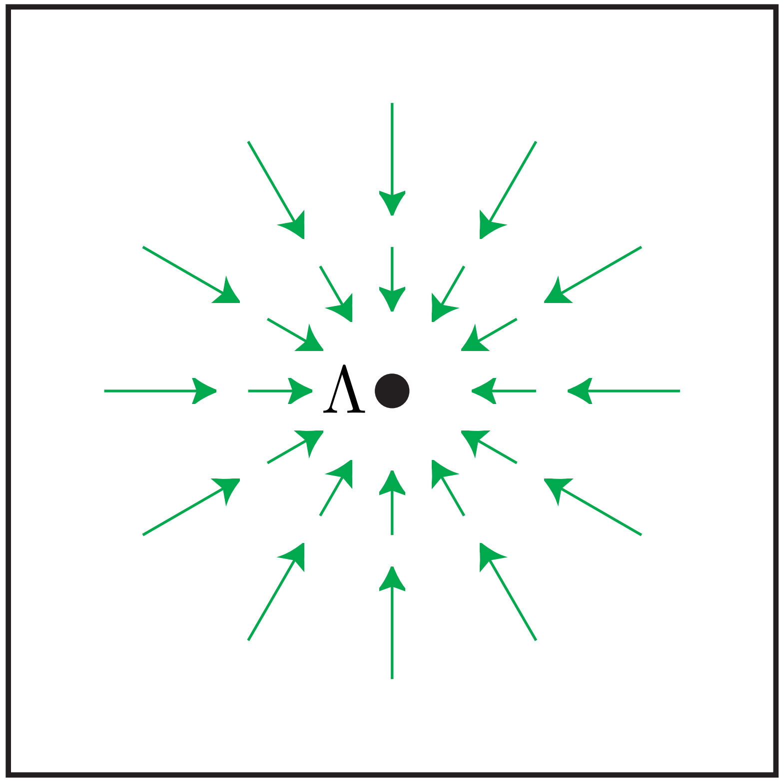
Attractor: $\Lambda \subset \mathcal{X}$

Where the flow goes at long times

(1) An invariant set

(2) A stable set: Perturbations off the set return to it

For example: Equilibrium

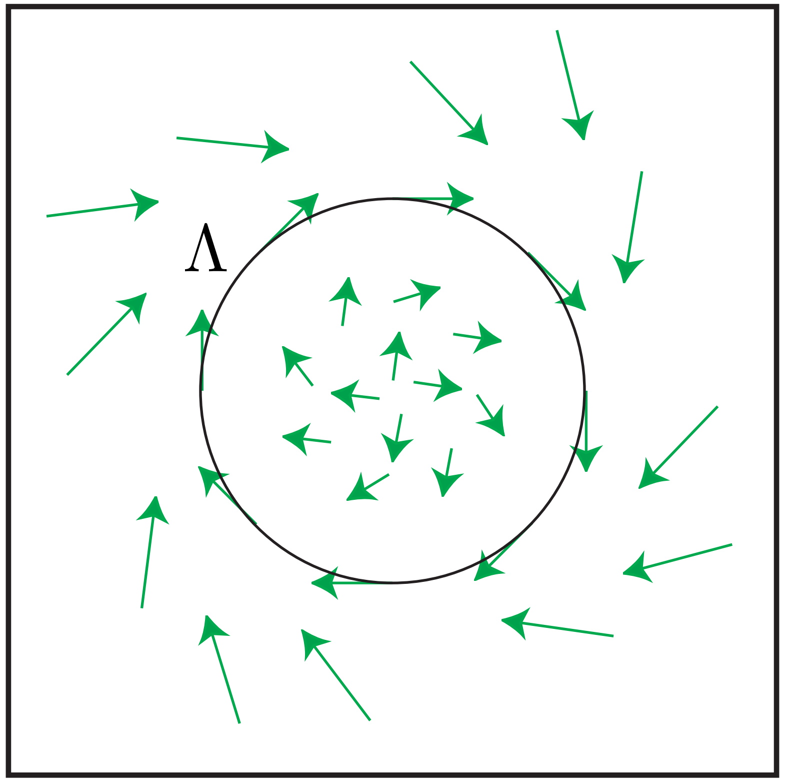


Stable Fixed Point

The Big Picture ...

Attractor: $\Lambda \subset \mathcal{X}$

For example: Stable oscillation



Limit Cycle

Note: Cycles in SHO, not stable in this sense; not attractors.

The Big Picture ...

Preceding:

A semi-local view ...

invariant sets and attractors in some region of the state space

Next:

A slightly Bigger Picture ...

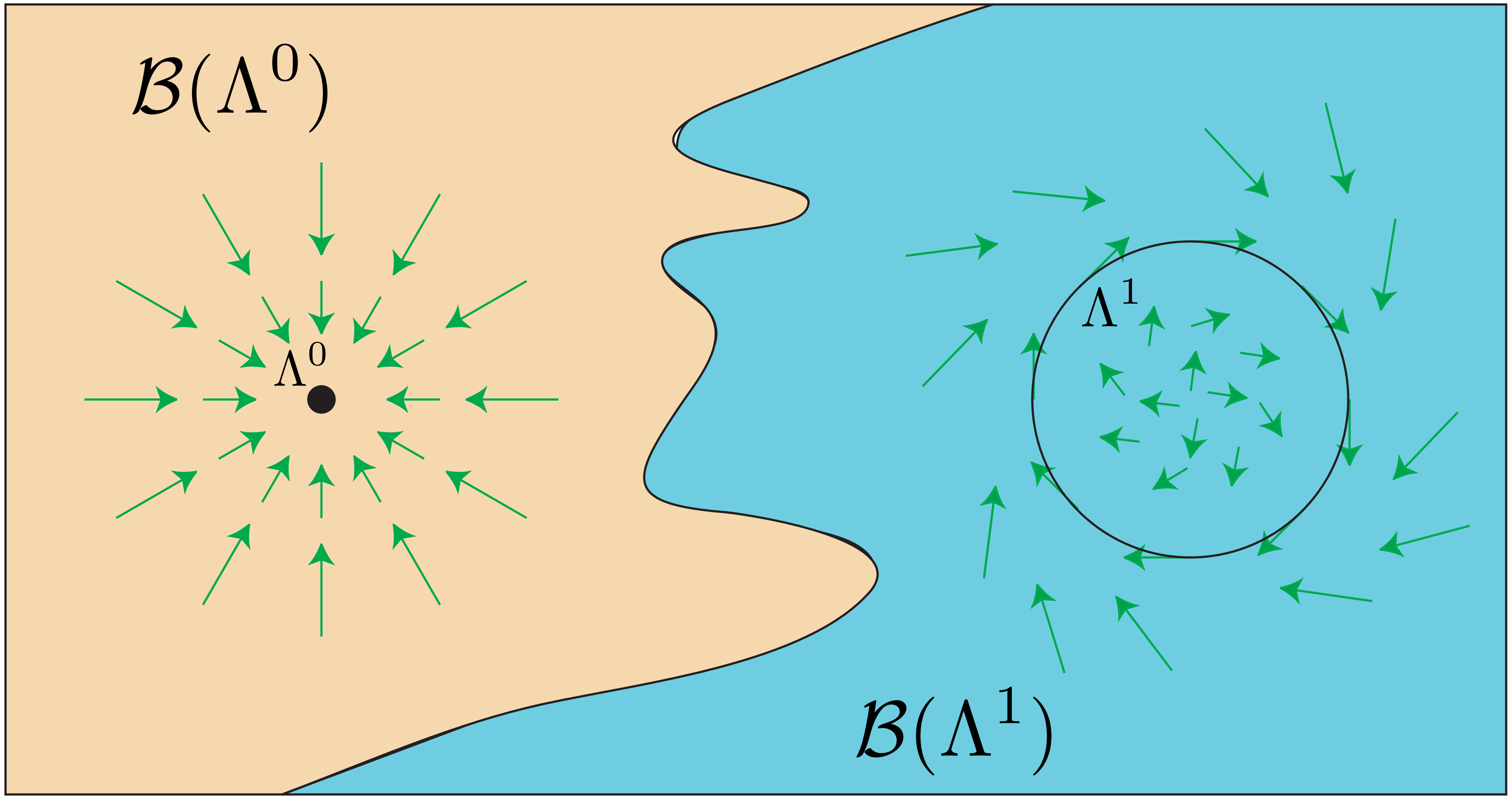
the full roadmap for the behavior of a dynamical system

The Big Picture ...

Basin of Attraction: $\mathcal{B}(\Lambda)$

The set of states that leads to an attractor Λ

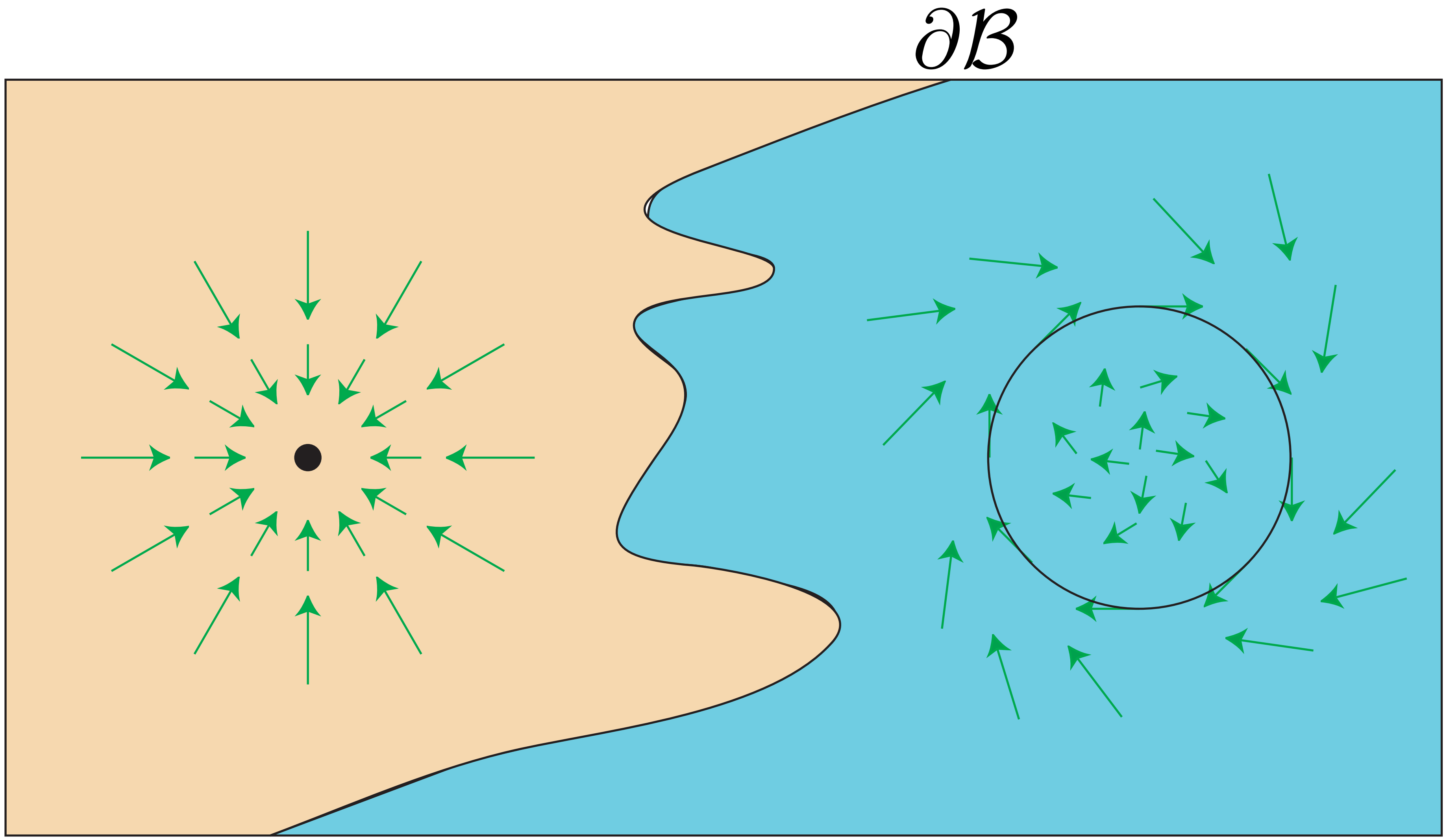
$$\mathcal{B}(\Lambda) = \{x \in \mathcal{X} : \lim_{t \to \infty} \phi_t(x) \in \Lambda\}$$



The Big Picture ...

Separatrix (aka Basin Boundary): $\partial \mathcal{B} = \mathcal{X} - \bigcup_i \mathcal{B}(\Lambda^i)$

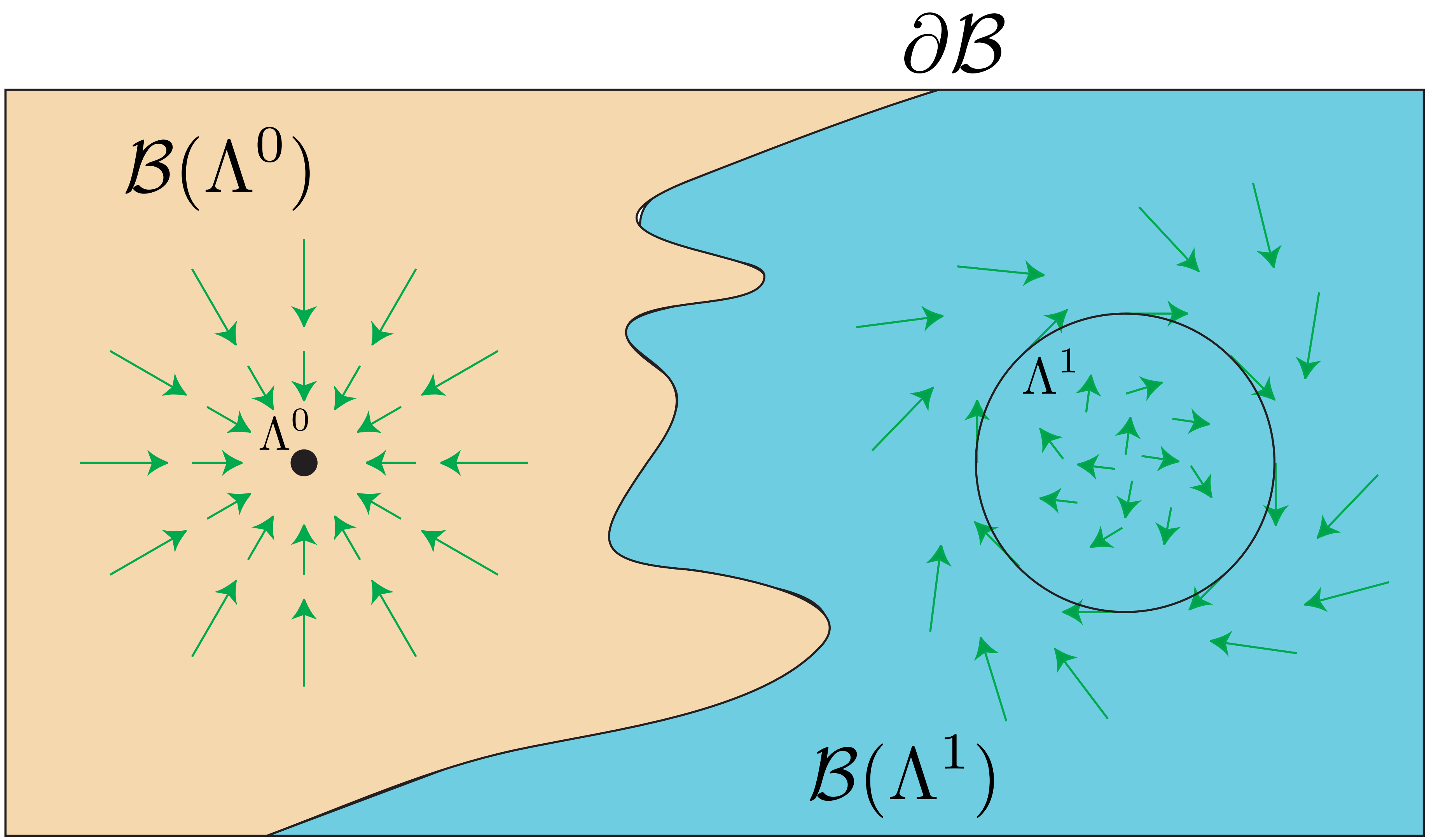
The set of states that do not go to an attractor
The set of states in no basin
The set of states dividing multiple basins



The Big Picture ...

The Attractor-Basin Portrait: $\Lambda^0, \Lambda^1, \mathcal{B}(\Lambda^0), \mathcal{B}(\Lambda^1), \partial\mathcal{B}(\Lambda^0)$

The collection of attractors, basins, and separatrices



The Big Picture ...

Back to the local, again

Submanifolds:

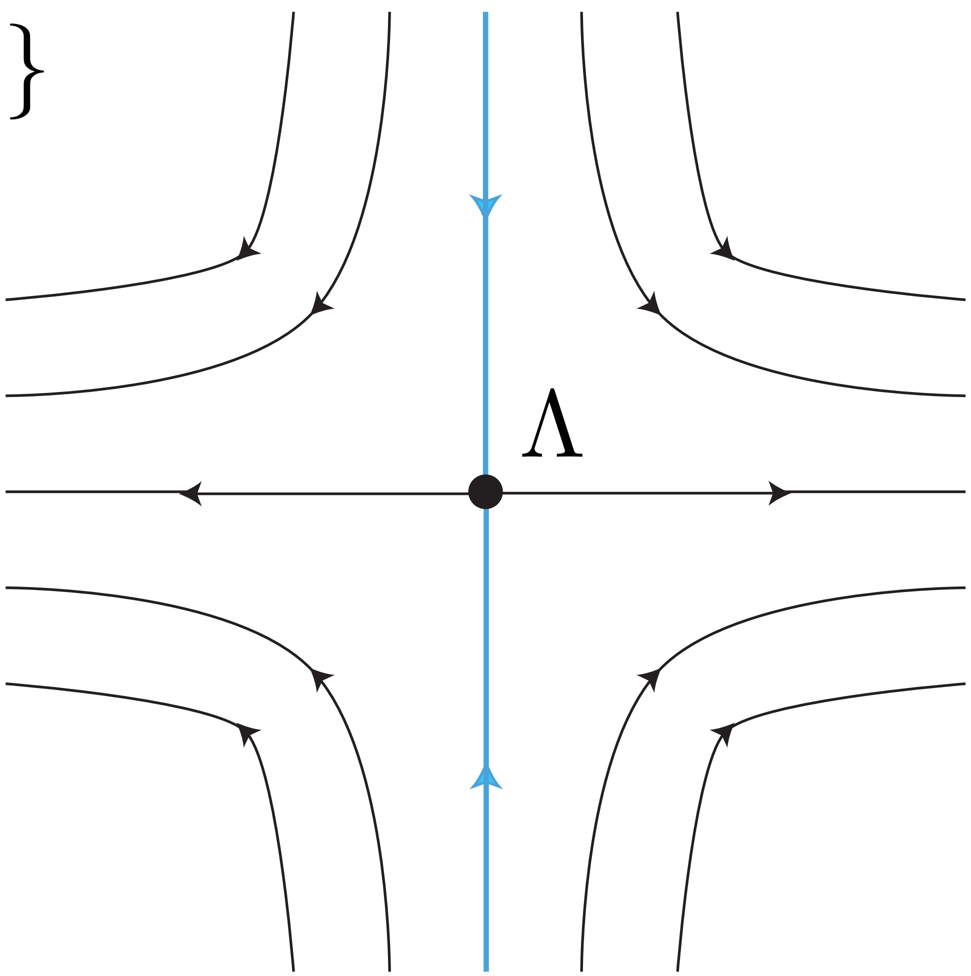
Split the state space into subspaces that track stability

Stable manifolds of an invariant set:

Points that go to the set

$$W^s(\Lambda) = \{x \in \mathcal{X} : \lim_{t \to \infty} \phi_t(x) \in \Lambda\}$$

$W^s(\Lambda)$

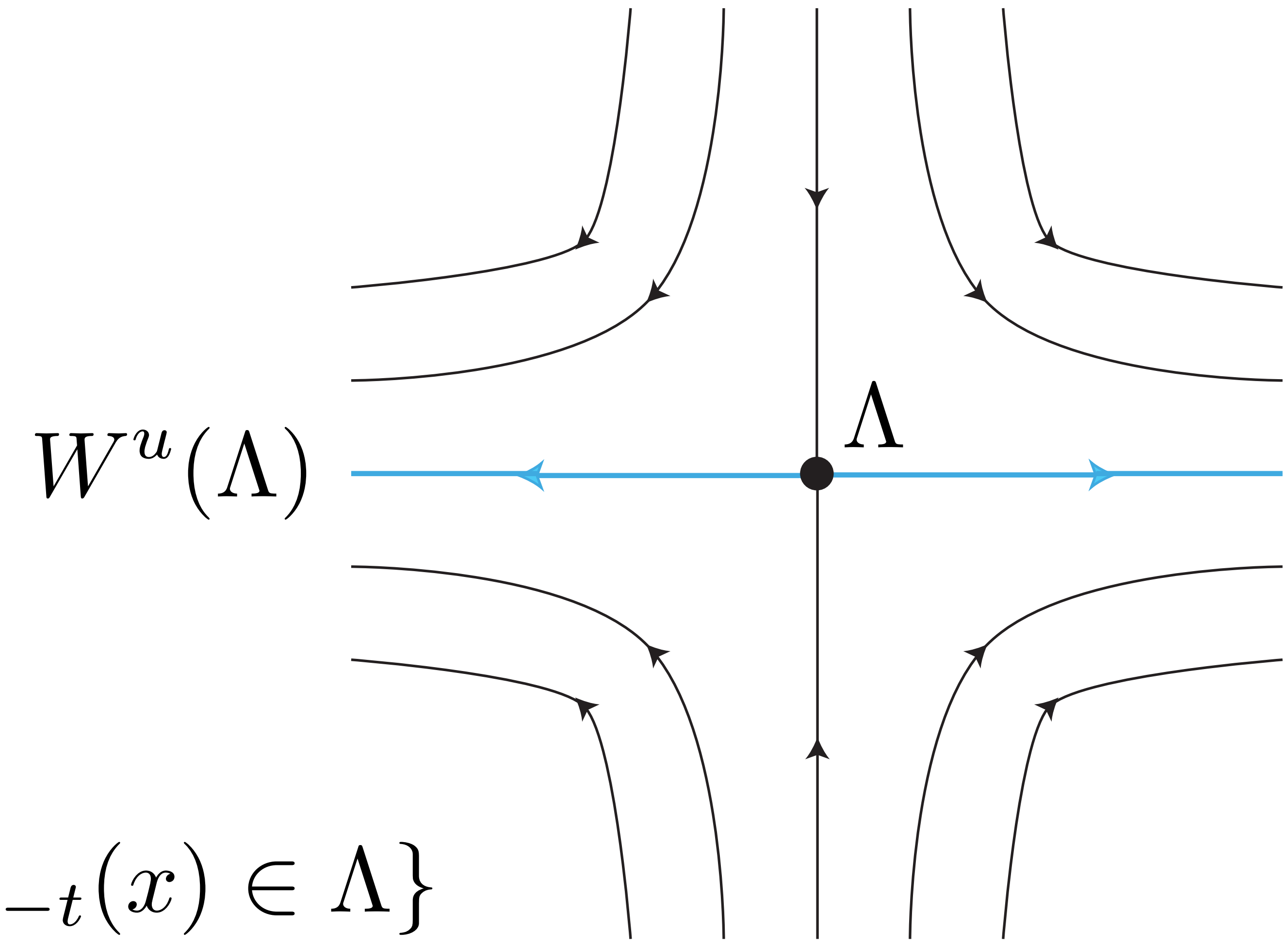


The Big Picture ...

Submanifolds ...

Unstable manifold:

Points that go to invariant set in *reverse time*



$$W^u(\Lambda) = \{x \in \mathcal{X} : \lim_{t \to \infty} \phi_{-t}(x) \in \Lambda\}$$

The Big Picture ...

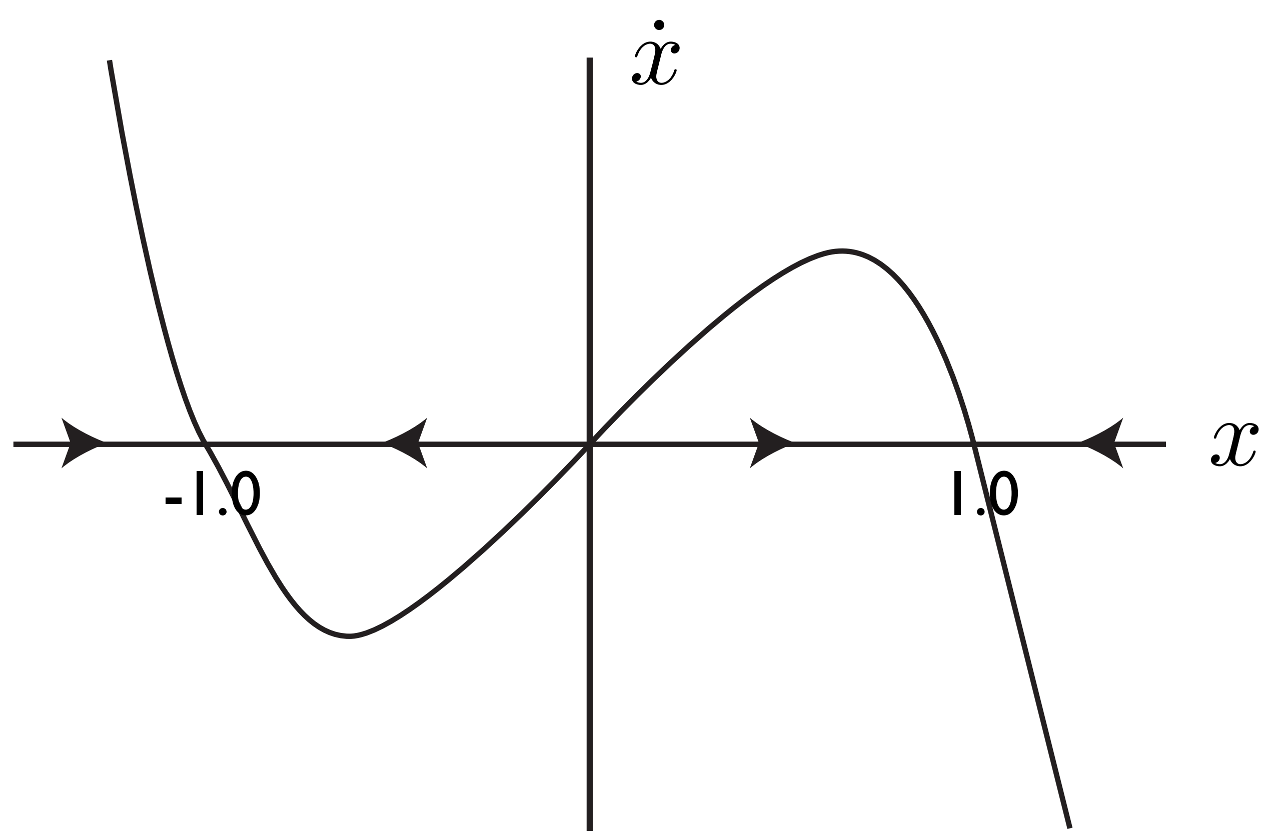
Example: 1D flows $\dot{x} = F(x)$

State Space: $\mathbf{R}$

State: $x \in \mathbf{R}$

Dynamic: $F : \mathbf{R} \to \mathbf{R}$

Flow: $x(T) = \phi_T\left(x(0)\right) = x(0) + \int_0^T dt\ F\left(x(t)\right)$



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The Big Picture ...

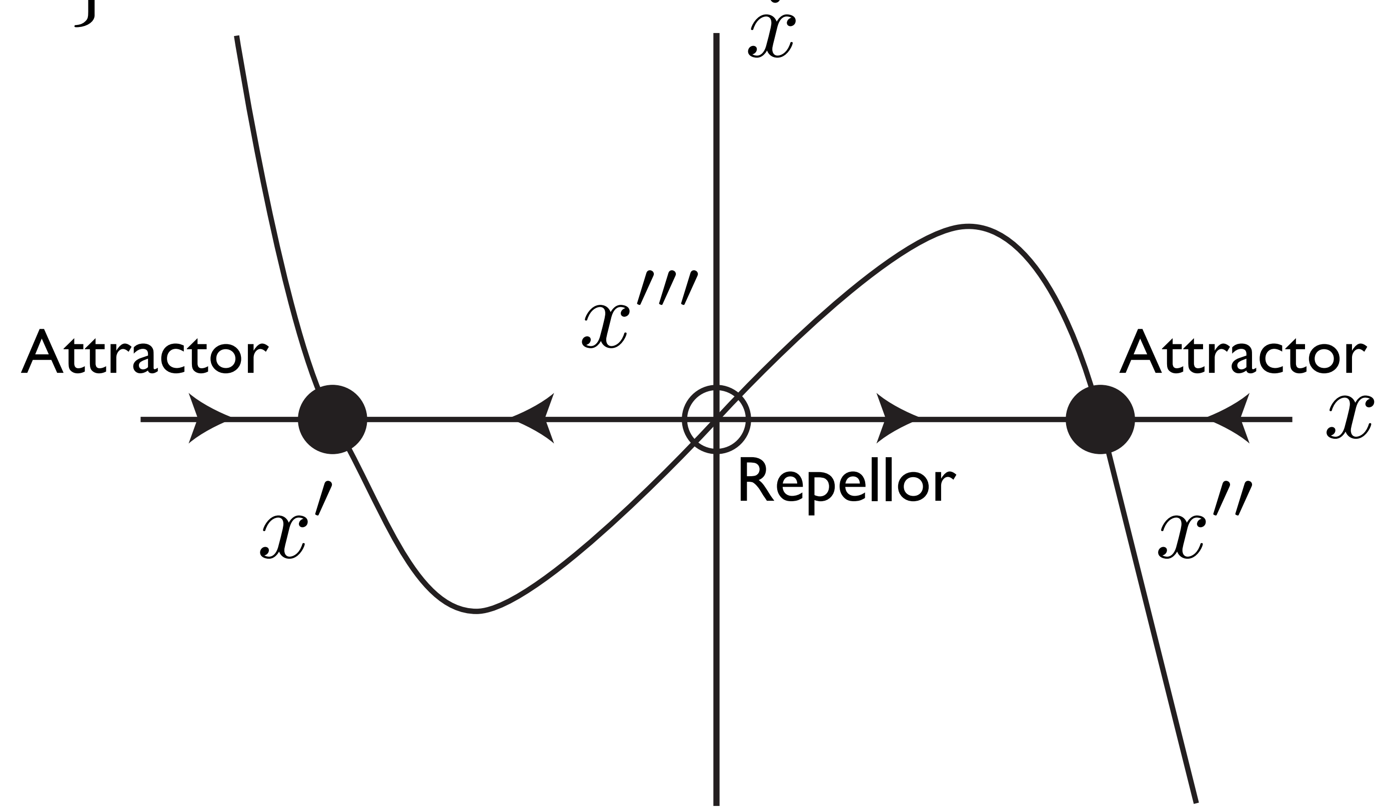
Example: 1D flows ...

Invariant sets: $\Lambda = \{x', x'', x'''\}$

Fixed points: $\dot{x} = 0$

Attractors: $x', x'' = \pm 1$

Repellor: $x''' = 0$

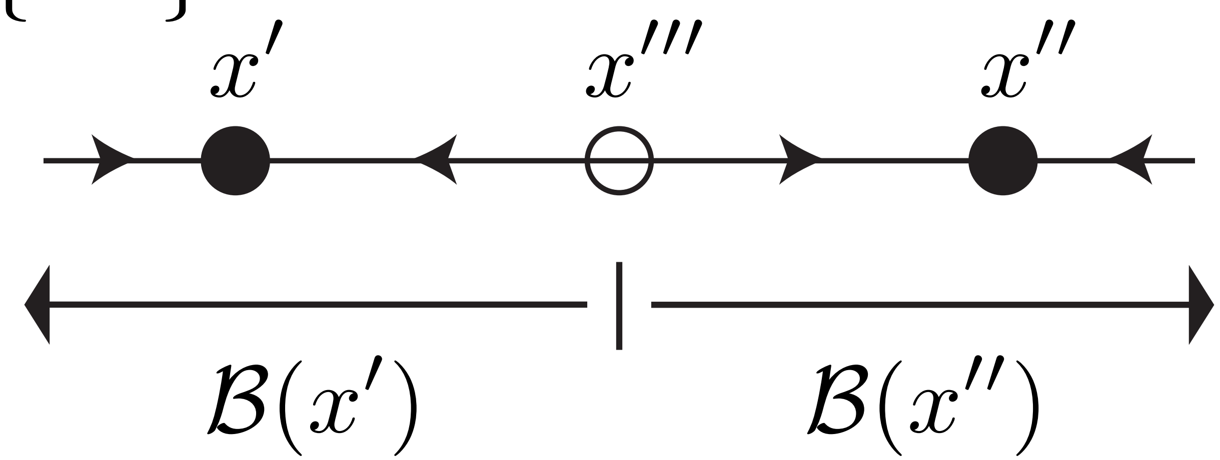


The Big Picture ...

Example: 1D flows ...

Basins: $\mathcal{B}(x') = [-\infty, 0)$
$\mathcal{B}(x'') = (0, \infty]$

Separatrix: $\partial\mathcal{B} = \{x'''\}$



Attractor-Basin Portrait:

$$x', x'', x''', \mathcal{B}(x'), \mathcal{B}(x''), \partial\mathcal{B}(x')$$

The Big Picture ...

Example: 1D flows ...

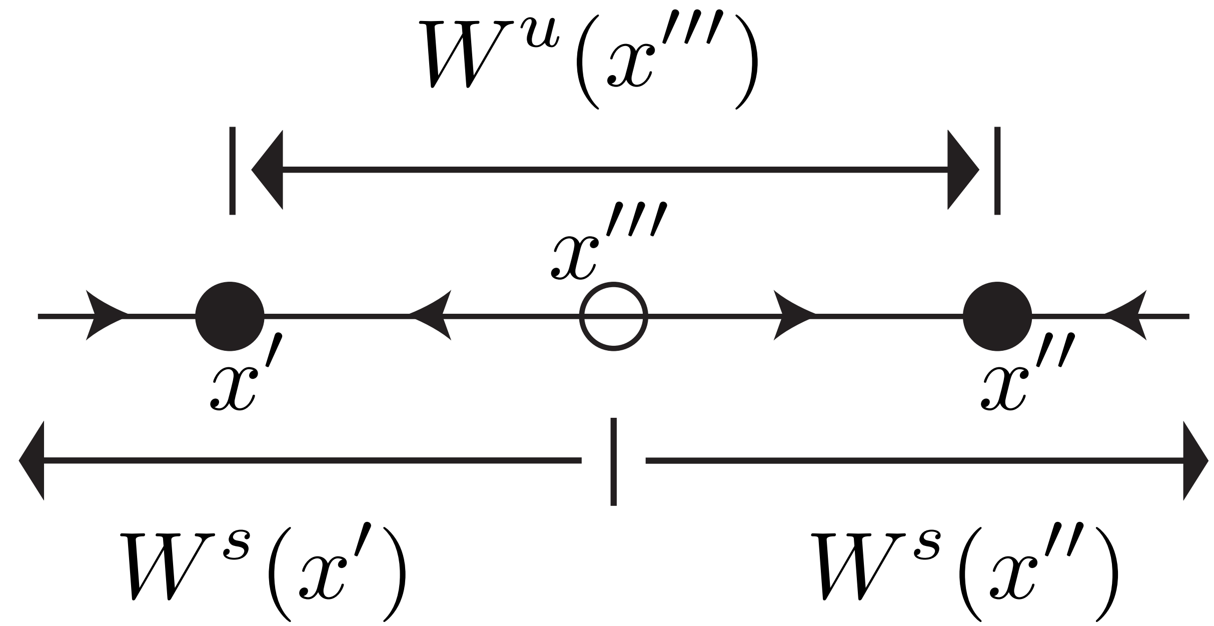
Stable manifolds:

$$W^s(x') = [-\infty, x''')$$

$$W^s(x'') = (x''', \infty]$$

Unstable manifold:

$$W^u(x''') = (x', x'')$$



The Big Picture ...

Example: 1D flows ...

Hey! Most of these dynamical systems are solvable!
For example, when the dynamic is polynomial
you can do the integral for the flow for all times.

What's the point of all this abstraction?

The Big Picture ...

Reading for next dynamics lecture:

NDAC, Sec. 6.0-6.4, 7.0-7.3, & 9.0-9.4

Reading for next programming lecture:

Python, Part II (Chapters 4 and 7-9) and Part III (Chapters 11-13).