Ergodic Theory and Dynamical Process Modeling

Foundations for Continuum Computational Mechanics

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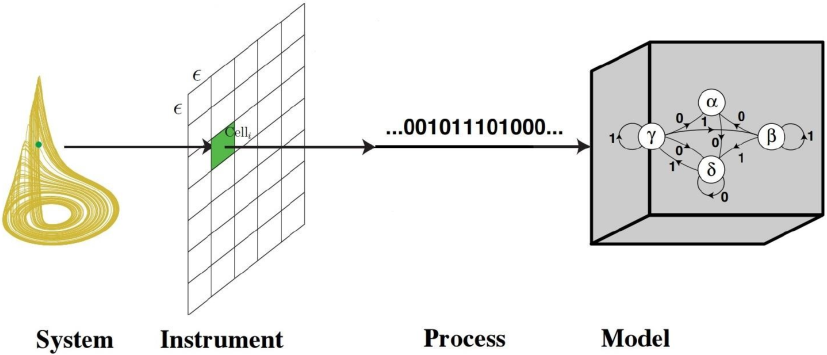
Nonlinear Modeling and Symbolic Processes

How does chaos generate randomness?

System	Observable	Measurement
$\omega_t \in \Omega$	$X : \Omega \to \mathcal{X}$	$x_t = X(\omega_t)$



Systematic and rigorous method for converting a continuous dynamical system into a fully-discrete, i.e. *symbolic*, stochastic process.

Nonlinear Modeling and Symbolic Processes

Symbolic processes are the objects traditionally modeled by computational mechanics,

Studied through *ϵ-machines*

Unique minimal sufficient statistic of past for predicting the future,
generated by *causal equivalence relation*:

$$\mathrm{past}_i \sim_\epsilon \mathrm{past}_j \iff \Pr(\mathrm{Future}|\mathrm{past}_i) = \Pr(\mathrm{Future}|\mathrm{past}_j)$$

- optimal prediction
- (causal) structure, organization
- directly calculate entropy rate
- process memory and complexity

Nonlinear Modeling and Symbolic Processes

When does nonlinear modeling work? – *generating partitions*

discrete-time dynamical system $(\Omega, \Phi : \Omega \to \Omega)$ – e.g. Poincare Map

$$\omega_{n+1} = \Phi(\omega_n)$$

partition phase space with *measurement function* $G_{\mathbb{P}} : \Omega \to \mathcal{A}$

$\mathbb{P}_i \cap \mathbb{P}_j = \emptyset$ and $\bigcup_{i=0}^{N} \mathbb{P}_i = \Omega$, and each partition carries unique symbol $a \in \mathcal{A}$

$\{\omega_0, \omega_1, \omega_2, \ldots\}$ becomes $\{a_0, a_1, a_2, \ldots\} = \{G_{\mathbb{P}}(\omega_0), G_{\mathbb{P}}\left(\Phi(\omega_0)\right), G_{\mathbb{P}}\left(\Phi^2(\omega_0)\right), \ldots\}$.

Nonlinear Modeling and Symbolic Processes

$G_{\mathbb{P}} \circ \Phi$ induces partition $\Phi^{-1}\mathbb{P}$ over Ω ; $(\Phi^{-1}\mathbb{P})_i$ is set of all $\omega \in \Omega$ s.t. $G_{\mathbb{P}}(\Phi(\omega)) \in \mathbb{P}_i$

each time step induces new $\Phi^{-n}\mathbb{P}$ whose elements are $\omega \in \Omega$ s.t. $G_{\mathbb{P}}(\Phi^n(\omega)) \in \mathbb{P}_i$

partition refinement $\mathbb{P} \vee \mathbb{Q} = \{\mathbb{P}_i \cap \mathbb{Q}_j \;:\; \mathbb{P}_i \in \mathbb{P} \text{ and } \mathbb{Q}_i \in \mathbb{Q}\}$ also a partition

first *dynamical refinement* of $\mathbb{P}$ under Φ is $\mathbb{P} \vee \Phi^{-1}\mathbb{P}$,
maps point $\omega \in \Omega$ to two-symbols $a_0 a_1 \in \mathcal{A} \times \mathcal{A}$

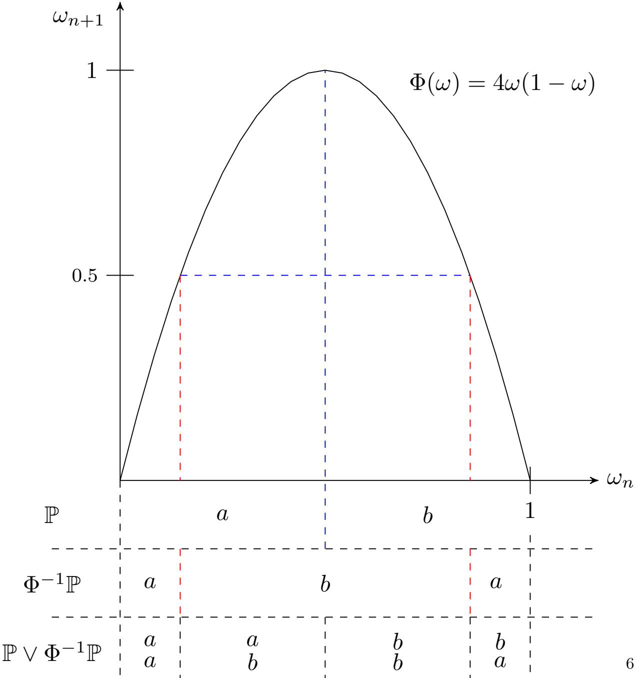
the *full dynamical refinement* of $\mathbb{P}$ under Φ is $\mathbb{P} \vee \Phi^{-1}\mathbb{P} \vee \Phi^{-2}\mathbb{P} \vee \Phi^{-3}\mathbb{P} \cdots$,
maps point $\omega \in \Omega$ to infinite-length symbol sequence $\mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \cdots$

a *generating partition* is a partition $\mathbb{P}$ s.t. the full dynamical refinement is a.e. one-to-one between points $\omega \in \Omega$ and infinite-length symbol sequences—volume of partition elements goes to 0 for full dynamical refinement

A.N. Kolmogorov, Russian Academy of Sciences (1959), Y.G. Sinai Russian Academy of Sciences (1959)

Generating Partition of Logistic Map

$$G_{\mathbb{P}}(x) = \begin{cases} a & 0 \leq \omega \leq 0.5 \\ b & 0.5 \leq \omega \leq 1 \end{cases}$$



Kolmogorov-Sinai Entropy: Randomness from Chaos

entropy of partition (in terms of invariant distribution over partition elements)

$$H(\mathbb{P}) = -\sum_i \Pr(\mathbb{P}_i) \log \Pr(\mathbb{P}_i)$$

entropy rate

$$h_\nu(\Phi, \mathbb{P}) = \lim_{N \to \infty} \frac{1}{N} H\left(\bigvee_{n=0}^{N} \Phi^{-n} \mathbb{P} \right)$$

Kolmogorov-Sinai (metric) entropy

$$h_\nu(\Phi) = \sup_{\mathbb{P}} h_\nu(\Phi, \mathbb{P})$$

achieved for generating partitions—provides variational principle for approximation

(asymptotic distribution over $\mathbb{P}_i$ limits to invariant density f^* of Φ for generating $\mathbb{P}$)

Pesin's theorem: relation to positive Lyapunov exponents

$$h_\nu(\Phi) \leq \sum_{\lambda_i > 0} \lambda_i$$

Y. Pesin. (1977) Russian Math Surveys

Nonlinear Modeling and Symbolic Processes

Symbolic process is a *shift dynamical system* $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ under *shift operator*

for two symbol sequence $x, y \in \mathcal{A}^{\mathbb{Z}}$, $y = \sigma(x) \iff y_i = x_{i+1}$ for all $i \in \mathbb{Z}$,
i.e. σ advances observation time index forward

$$
\begin{tikzcd}
\underline{\text{System}} & \underline{\text{Measurement Symbol}} & \underline{\text{Symbol Sequence}} \\
\omega_n \in \Omega \arrow[r, "G_{\mathbb{P}} : \Omega \to \mathcal{A}"] \arrow[d, "\Phi"] & x_0 \in \mathcal{A} & \cdots x_{-2} x_{-1} x_0 x_1 x_2 \cdots \arrow[l, "\mathbf{0}"'] \arrow[d, "\sigma"'] \\
\omega_{n+1} = \Phi(\omega_n) \arrow[r, "G_{\mathbb{P}} : \Omega \to \mathcal{A}"] & y_0 = x_1 \in \mathcal{A} & \cdots y_{-2} y_{-1} y_0 y_1 y_2 \cdots \arrow[l, "\mathbf{0}"']
\end{tikzcd}
$$

Converted nonlinear dynamical system into (symbolic) measurement process governed by **linear*, infinite-dimensional operator* σ that advances observation time

D. Lind and B. Marcus (1995) Cambridge University Press

Systems, Data, and Models

Advantages of nonlinear modeling:

- symbolic processes allow for discrete information and computation theory
- clean and rigorous framework for information storage, generation, and processing
- interpretability of system structure and organization through ϵ-machines

Challenges:

- even for idealized systems, generating partitions hard to find (e.g. Henon map)
- for a given physical system, don't have full control of system measurements
- scalability of inference and interpretability for large alphabets

generalize to continuum setting using Koopman and Perron-Frobenius Operators

A. Lasota and M. C. Mackey (2013). Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics. Springer

System: $\omega_0 \in \Omega$

Observations: $x_0 = X(\omega_0)$

$$
\begin{tikzcd}
\Omega \arrow[rr, "X : \Omega \to \mathcal{X}"] \arrow[d, "\Phi^t"] \arrow[drr, "X_t : \Omega \to \mathcal{X} = U^t X = X \circ \Phi^t"] & & \mathcal{X} \arrow[d, "U^t"] \\
\Omega \arrow[rr, "X : \Omega \to \mathcal{X}"] & & \mathcal{X}
\end{tikzcd}
$$

Credit: NASA–Apollo 17 crew

Systems and Data

The "true" physical system described by dynamical system $(\Omega, \Sigma_\Omega, \nu, \Phi)$:

- phase space Ω is complete metric space (typically $\mathbb{R}^d$ or d-dimensional manifold)
- Σ_Ω a σ-algebra (Borel sets)
- ν a reference measure (Borel or Lebesgue)—phase space volume
- Φ the generator of (semi)group of measurable flow maps $\{\Phi^t : \Omega \to \Omega\}$

$$\Phi = \lim_{\tau \to 0} \frac{1}{\tau}\left(\Phi^t(\omega) - \Phi^{t+\tau}(\omega)\right)$$

- orbits $\{\omega(t) \ : \ t \in \mathbb{R}_{(\geq 0)}\}$ continuous in time
- discrete intervals are bounded $||\Phi^{t+\delta}(\omega_0) - \Phi^t(\omega_0)|| < \epsilon$

Systems and Data

The "observed" or "measured" system is measurable space $(\mathcal{X}, \Sigma_{\mathcal{X}})$

Observable $x \in \mathcal{X}$ generated by the dynamical system under the measurable mapping $X : \Omega \to \mathcal{X}$ so that $x_t = X(\omega_t)$

Generally interested in *partially-observable systems*, for which X is *not invertible*: knowledge of x insufficient for determining state ω of the true system
$\implies$ there are "unobservable" or "immeasureable" degrees of freedom in ω

Will later consider a second set of observables $(\mathcal{Y}, \Sigma_{\mathcal{Y}})$ that are also given as a (generally non-invertible) measurable map $Y : \Omega \to \mathcal{Y}$ s.t. $y_t = Y(\omega_t)$.

X (and Y) represent "windows" through which we can view true physical system, but can never have a full view with $\mathcal{X} = \Omega$

Asymptotic behavior of ω may be reconstructable from x: *delay-coordinate embedding*

N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw (1980) F. Takens (1981)

Koopman Operators

system "observable" $f : \Omega \to \mathbb{C}$, element of a function space (typically $L^\infty(\Omega, \nu)$ or $L^2(\Omega, \nu)$)

Koopman operators $\{U^t : \mathcal{F} \to \mathcal{F}\}$ evolve observables through composition with Φ^t

$$U^t f = f \circ \Phi^t$$
$$f_t(\omega) \equiv f\big(\Phi^t(\omega)\big) = U^t f(\omega)$$

linear, infinite-dimensional operators whose action on observable $f \in \mathcal{F}$ gives the time shifted observable (function) $f_t = U^t f$

Inherits (semi)group structure of $\{\Phi^t\}$ s.t. $U^t \circ U^{\Delta t} = U^{t+\Delta t}$ and generated by

$$U f = \lim_{t \to 0} \frac{1}{t}(U^t f - f)$$

X (and Y) vector-valued observables, i.e. $X_i = f$, evolved by product operator U^t

Dynamical Processes

observables may be collected in a *time series* $\{x_0, x_1, \ldots, x_{T-1}\}$—a time-ordered sequences of "measurements" of x taken at uniform time intervals.

a *dynamical process* is bi-infinite time series of observables $\{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$

* for non-invertible dynamics, index is "observation time"

in terms of Koopman operators:

$$x_t = X(\omega_t) = X\big(\Phi^t(\omega_0)\big) = X_t(\omega_0) = U^t X(\omega_0)$$

Goal of dynamical process modeling: infer or approximate action of U on observables X

$$\{x_0, x_1, x_2, x_3, \ldots\} = \{U^0 X(\omega_0), U^1 X(\omega_0), U^2 X(\omega_0), U^3 X(\omega_0)\}$$

(finite) *reconstructability*: X in finite invariant subspace $\mathbf{U}$ s.t. $U^t X \in \mathbf{U}$ for all t

S. L. Brunton, B. W. Brunton, J. L. Proctor, and J. N. Kutz (2016). PloS One

Perron-Frobenius Operators

Rather than evolve observables, *Perron-Frobenius operators* $P^t : L^1(\Omega, \nu) \to L^1(\Omega, \nu)$ evolve *densities* $f \in L^1(\Omega, \nu)$: $f \geq 0$ and $||f|| = 1$.

$$f_t = P^t f$$

$$\int_S P^t f d\nu = \int_{\left(\Phi^t\right)^{-1}(S)} f d\nu \qquad \text{for } S \in \Sigma_\Omega$$

May also consider P^t evolving $L^2(\Omega, \nu)$ measures μ: $\mu_t = P^t \mu$

relation to densities through Radon-Nikodym:

$$\mu^f(S) = \int_S f d\nu \quad \text{and} \quad f = \frac{d\mu^f}{d\nu}$$

$P^t : L^1 \to L^1$ adjoint of $U^t : L^\infty \to L^\infty$ and $P^t : L^2 \to L^2$ adjoint of $U^t : L^2 \to L^2$

$$\langle P^t f, g \rangle = \langle f, U^t g \rangle$$

Stochastic Processes

Can now formulate *continuous stochastic processes* generated by dynamical systems

- dynamical system $(\Omega, \Sigma_\Omega, \nu, \Phi)$ with initial probability density (ensemble) f_0
- density at time t is $f_t = P^t f_0 \implies$ probability measure $\mu_t(S) = \int_S f_t d\nu$
- at time t have probability space $(\Omega, \Sigma_\Omega, \mu_t) \implies$ observable map $X : \omega_t \mapsto x_t$ now defines random variable X_t
- X_t distributed according via pushforward $\mu_t^X(S_\mathcal{X}) = \mu_t(X^{-1}(S_\mathcal{X}))$ for $S_\mathcal{X} \in \Sigma_\mathcal{X}$

$$\mathrm{pr}(X_t \in S_\mathcal{X}) = \int_{S_\mathcal{X}} d\mu_t^X = \int_{X^{-1}(S_\mathcal{X})} d\mu_t = \int_{X^{-1}(S_\mathcal{X})} f_t \, d\nu = \int_{X^{-1}(S_\mathcal{X})} P^t f_0 \, d\nu$$

Therefore, an initial density f_0 on a dynamical system $(\Omega, \Sigma_\Omega, \nu, \Phi)$ produces continuous stochastic process $\{X_0, X_1, X_2, \dots\}$

random variables are actually $X(t, \omega)$; for fixed ω a *sample path* given by $t \mapsto X(t, \omega)$,
here are (continuous) dynamical processes $\{x_0, x_1, x_2, \dots\}$

Connection to Symbolic Processes: Koopman

symbolic process:

$$\{\omega_0, \omega_1, \omega_2, \dots\} \to \{a_0, a_1, a_2, \dots\} = \{G_{\mathbb{P}}(\omega_0), G_{\mathbb{P}}(\Phi(\omega_0)), G_{\mathbb{P}}(\Phi^2(\omega_0)), \dots\}$$

dynamical process:

$$\{\omega_0, \omega_1, \omega_2, \dots\} \to \{x_0, x_1, x_2, \dots\} = \{U^0 X(\omega_0), U^1 X(\omega_0), U^2 X(\omega_0), \dots\}$$

Logistic map:

Partition isomorphic to generating partition given by

$$G_{\mathbb{P}}(x) = \begin{cases} 0 & 0 \le x \le 0.5 \\ 1 & 0.5 \le x \le 1 \end{cases}$$

$G_{\mathbb{P}}(x)$ generally given as sum of labeled indicator functions for partition elements $\mathbb{P}_i$:

$G_{\mathbb{P}}(x) = \sum_{i=0}^{N-1} i \mathbb{1}_{\mathbb{P}_i}, \quad N = ||\mathbb{P}||$

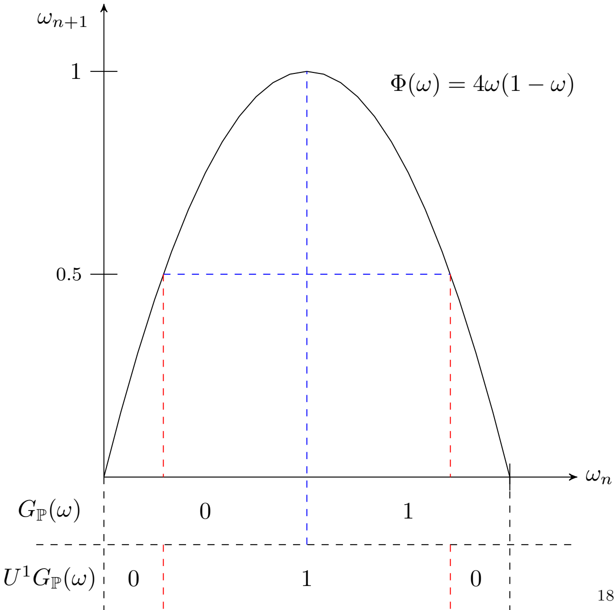
$$\mathbb{1}_{\mathbb{P}_i} = \begin{cases} 1 & x \in \mathbb{P}_i \\ 0 & x \notin \mathbb{P}_i \end{cases}$$

Connection to Symbolic Processes: Koopman

$$G_{\mathbb{P}}(\omega) = \begin{cases} 0 & 0 \le \omega \le 0.5 \\ 1 & 0.5 \le \omega \le 1 \end{cases}$$

$$U^1 G_{\mathbb{P}}(\omega) = G_{\mathbb{P}}\big(\Phi(\omega)\big) = \Phi^{-1}\mathbb{P}$$

$$U^1 G_{\mathbb{P}}(\omega) = \begin{cases} 1 & \frac{1-\sqrt{\frac{1}{2}}}{2} \le \omega \le \frac{1+\sqrt{\frac{1}{2}}}{2} \\ 0 & \text{otherwise} \end{cases}$$



Connection to Symbolic Processes: Perron-Frobenius

reference measure vs invariant measure

if reference measure is invariant, constant density $f = 1$ is stationary: $P^t 1 = 1$

for logistic map with Borel / Lebesgue reference measure:

$$P^1 1 = \frac{1}{2\sqrt{1-\omega}}$$

$\implies$ Borel measure not invariant

invariant measure μ^* found from solving $P^1 f^* = f^*$:

$$\mu^*(S) = \int_S f^* d\omega = \int_S \frac{d\omega}{\pi\sqrt{\omega(1-\omega)}}$$

Ulam and von Neumann (1947)

Connection to Symbolic Processes: Perron-Frobenius

reference measure vs invariant measure

consider uniform density on $\mathbb{P}_1$ $\big(\Pr(a_0 = 1) = 1\big)$ and its evolved density

$$f(\omega) = \begin{cases} 0 & 0 \leq \omega \leq 0.5 \\ 2 & 0.5 \leq \omega \leq 1 \end{cases} \qquad P^1 f(\omega) = \frac{1}{2\sqrt{1-\omega}}$$

with Borel reference measure:

$$\Pr(a_1 = 1) = \quad 2\int_{\frac{1}{2}}^{\frac{1+\sqrt{\frac{1}{2}}}{2}} d\omega = \int_{\frac{1}{2}}^{1} \frac{d\omega}{2\sqrt{1-\omega}} = \sqrt{\frac{1}{2}}$$

with invariant reference measure:

$$\Pr(a_1 = 1) = \quad 2\int_{\frac{1}{2}}^{\frac{1+\sqrt{\frac{1}{2}}}{2}} d\mu^* = 2\int_{\frac{1}{2}}^{\frac{1+\sqrt{\frac{1}{2}}}{2}} \frac{d\omega}{\pi\sqrt{\omega(1-\omega)}} = \frac{1}{2}$$

Ergodicity

Typical to consider *measure-preserving* dynamical system $(\Omega, \Sigma_\Omega, \nu, \Phi)$ with $\Phi\nu = \nu$

Assumes a *stationary* stochastic process: $\mathrm{pr}(X_t \in S_\mathcal{X}) = \mathrm{pr}(X_0 \in S_\mathcal{X})$ for all t

This corresponds to dynamics on an *attractor*, but want to consider more general dynamics that include transient relaxation to the attractor

We do this using *ergodic components*, which correspond to *basins of attraction* (including the attractor itself)

A dynamical system $(\Omega, \Sigma_\Omega, \nu, \Phi)$ is *ergodic* if $\nu(S) = 0$ or $\nu(S) = 1$ for every *invariant set* $S : \Phi^{-1}(S) = S$
$\implies$ all invariant sets are trivial subsets of Ω and we must study Φ on the entire space Ω

*** Note ergodicity is independent of measure preservation—but they are often assumed together

Attractors and Basins

invariant set : $\Phi^{-1}(S) = S$
forward-invariant set : $\omega_0 \in S \implies \Phi^t(\omega_0) \in S$ for all t

an invariant set is necessarily forward-invariant, but converse not true

An *attractor* of $(\Omega, \Sigma_\Omega, \nu, \Phi)$ is a set $A \in \Sigma_\Omega$ with

- A is a forward-invariant set of Ω under Φ
- there exists an open set $B \supset A$, call the *basin of attraction* of A, s.t. for every $\omega \in B$ $\lim_{t\to\infty} \Phi^t(\omega) \in A$, and
- there is no proper subset of A with the first two properties

Attractors are *not* Φ-invariant, but *basins are Φ-invariant*

Can define basins as limit of pre-images of attractor: $B = \lim_{t\to\infty} \left(\Phi^t\right)^{-1} A$

J. D. Meiss (2007) *Differnetial Dynamical Systems*, SIAM

Attractor Basins as Ergodic Components

Multi-stable dynamical systems with multiple attractors can be partitioned: each basin may be treated independently since, by definition, orbits never cross basin boundaries

Without loss of generality, will from here out consider systems that either:

- have one attractor with Ω as basin
- independently consider *ergodic components*: reduced systems with $\Omega = B$

System $(\Omega, \Sigma_\Omega, \nu, \Phi)$ considered this way is thus *ergodic*

Asymptotic behavior on the attractor:

- ergodicity guarantees P^t has unique invariant density $f^* : P^t f^* = f^*$
- this defines *asymptotic invariant measure* $\mu^*(S) = \int_S f^* d\nu$
- ergodic theorem: time averages equal phase space averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(\Phi^k(\omega)\right) = \frac{1}{\mu^*(\Omega)} \int_\Omega f(\omega) d\mu^*$$

The Prediction Problem

We set up a general prediction problem for dynamical processes as follows:

For a physical system $(\Omega, \Sigma_\Omega, \nu, \Phi)$ let the observable $X : \Omega \to \mathcal{X}$ represent the collection of all possible measurements that can be made of the system.

Let $Y : \Omega \to \mathcal{Y}$ represent "variables of interest"

* typically $\mathcal{Y} \subseteq \mathcal{X}$, but variables of interest may not be adequately measurable, in which case $\mathcal{Y} \nsubseteq \mathcal{X}$
* variables in $\mathcal{X}$ that are not in $\mathcal{Y}$ are sometimes called *exogeneous*

From definitions / assumptions, the physical system $(\Omega, \Sigma_\Omega, \nu, \Phi)$ generates the *true* dynamical processes (indices again correspond to observation time)

$$\{\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\}$$
$$\{\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots\}$$

The Prediction Problem

Denote *past* random variable $\overleftarrow{X} = \{\ldots, X_{-2}, X_{-1}, X_0\}$ with realizations $\overleftarrow{x} = \{\ldots, x_{-2}, x_{-1}, x_0\}$ represent observed series of measurements up to present time t_0

Similarly denote *future* random variable $\overrightarrow{Y}_\tau = \{Y_1, Y_2, Y_3, \ldots, y_\tau\}$ with realizations $\overrightarrow{y}_\tau = \{y_1, y_2, y_3, \ldots, y_\tau\}$ represent future values of variables of interest to lead time τ

deterministic prediction problem: find *target function* $\mathcal{T}_\tau$ that maps $\overleftarrow{x}$ to $\overrightarrow{y}_\tau$

$$\text{minimize} \;\; ||\mathcal{T}_\tau \circ \overleftarrow{x} - U^\tau Y||_{L^2(\nu)}$$

R. Alexander and D. Giannakis, Physica D: Nonlinear Phenomena (2020)

probabilistic prediction problem: find *conditional distribution* $\mathrm{pr}(\overrightarrow{Y}_\tau | \overleftarrow{X} = \overleftarrow{x})$

Predictive Models of Dynamical Processes

Model dichotomy:

myopic models—learn target functions for finite τ (and finite-length pasts)

- (N)ARIMA(X)
- Analogue forecasting
- Neural networks (e.g. LSTM, TCN, RC, etc.)

process models—generative models that approximate U (i.e. $\lim \tau \to \infty$)

- approximate action of U with surrogate dynamical system $\widetilde{\Phi}_\alpha : \mathcal{X} \to \mathcal{X}$
- numerical simulations with discretization X and approximation scheme α
- complex physics simulations with analysis/assimilation $\widetilde{X}$ from measurements X with parameterizations α
- reduced-order models (ROMS)
- (generalized) Galerkin approximations of U

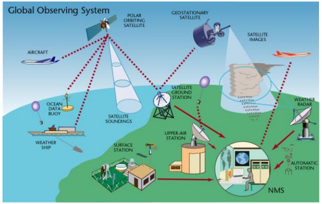
Example: Earth System

(analysis / assimilation)

System

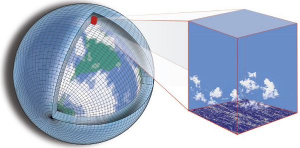
$X : \Omega \to \mathcal{X}$

Observations



$\widetilde{X}$

Simulation



“data image”

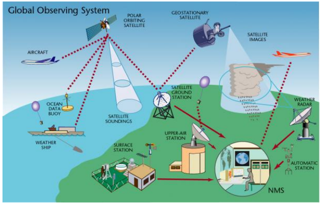
Φ^t

$X_t = U^t X$

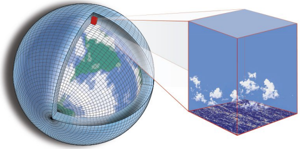
U^t

$\widetilde{\Phi}^t_\alpha$

$X : \Omega \to \mathcal{X}$



$\widetilde{X}$



Credit: NASA–Apollo 17 crew

Credit: WMO

Credit: Tapio Schneider/Kyle Pressel/Momme Hell/Caltech

Galerkin Approximations

EDMD: project U^t onto finite subspace spanned by dictionary $\mathbf{\Psi} = [\psi^1, \ldots, \psi^k]^{\mathrm{T}}$

System | Observations | Galerkin Approximation

$$
\begin{tikzcd}
\omega_0 \in \Omega \arrow[r, "X : \Omega \to \mathcal{X}"] \arrow[dd, "\Phi^t"] \arrow[ddr, "X_t = U^t X"] & x_0 = X(\omega_0) \arrow[r, "\mathbf{\Psi}"] \arrow[dd, "U^t"'] & {[\psi^1(x_0), \ldots, \psi^k(x_0)]^{\mathrm{T}}} \arrow[d, "\mathbf{U}^t"'] \\
 & & {[\mathbf{U}^t\psi^1(x_0), \ldots, \mathbf{U}^t\psi^k(x_0)]^{\mathrm{T}}} \arrow[d, dashed, leftrightarrow, "\text{minimize}"] \\
\omega_t = \Phi^t(\omega_0) \arrow[r, "X : \Omega \to \mathcal{X}"] & x_t = X(\omega_t) \arrow[r, "\mathbf{\Psi}"] & {[\psi^1(x_t), \ldots, \psi^k(x_t)]^{\mathrm{T}}}
\end{tikzcd}
$$

M. O. Williams, I. G. Kevrekidis, and C. W. Rowley (2015). Journal of Nonlinear Science
S. Klus, P. Koltai, and C. Schütte (2016). Journal of Computational Dynamics