University of California, Davis
Physics Department

PHY 2^8 - Natural Computation and Self-Organization
Final Project

Interplanetary Transport Network

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Abstract

The Interplanetary Transport Network is a collection of gravitationally determined pathways through the solar system that require very little energy for an object to follow. The ITN makes particular use of Lagrange points as locations where trajectories through space can be redirected using little or no energy. These points have the peculiar property of allowing objects to orbit around them, despite the absence of any material object therein.
This short paper is a review of some important applications of this technique, such as the use of the three-body problem and the four-body problem to identify the best trajectory with respect to fuel savings. It will be shown that in the Earth-to-Moon path it is possible to save up to 30% of the fuel required by the usual old-fashioned path.

1 Introduction and background

The study of some of the possible natural trajectories in the solar system is studied. The general case is reduced, when possible, to a three body-problem, a system extensively studied in the literature. The three or more body problem is a complicated nonlinear mathematical system which generates paths which can seem unlikely or odd to the non-expert eye.
To fully understand this short review it is highly suggested to be familiar with Lagrangian and Hamiltonian systems, with a major focus on the concept of the five Lagrangian points in the three-body problem.

2 Dynamical System

2.1 The three-body problem and the simplified four-body problem

The procedure involves a study of the three-body problem and the possible application of two combined three-body-problem-solutions to solve the four-body problem in certain domains. I.e., to reduce a specific four-body problem, when possible, to two, previously and generally solved, three-body problems.

2.2 Equations of motion

The planar restricted three-body problem can be written as the following system, assuming two massive bodies orbiting around each other and a third "light" body orbiting in the effective potential created by the two massive objects.

$$\dot{x} = \frac{\partial H}{\partial p_x} = p_x + y \tag{2.1}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = p_y - x \tag{2.2}$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = p_y - x - \overline{U}_x \tag{2.3}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -p_x - y - \overline{U}_y \tag{2.4}$$

with $\overline{U}(x, y)$ being

$$\overline{U}(x, y) = -\frac{1}{2}\left(\mu_1 r_1^2 + \mu_2 r_2^2\right) - \frac{\mu_1}{r_1} - \frac{\mu_2}{r_2} \tag{2.5}$$

where μ_1 and μ_2 are quantities representing the relative distance from the center of mass of the two massive bodies and r_1 and r_2 the distances of the third light body from the two massive bodies. (x, y) are the coordinates of the light body in the rotating two-body system where the two massive bodies are at rest. The potential $\overline{U}(x, y)$ results to be time independent and works as an effective potential for the light body.
This system admits as solutions different classes of trajectories, which will be described in the next paragraph.

2.3 Description of the classes of trajectories

The shape of the effective potential can be shown in a three-dimensional graph, where the maxima, minima and saddle points represent the points around which local stable or unstable orbits exist. The Lagrangian points known as L_3, L_4 and L_5, also very important because of some peculiar stability properties, will not be discussed in this review, which focuses on the points L_1 and L_2.

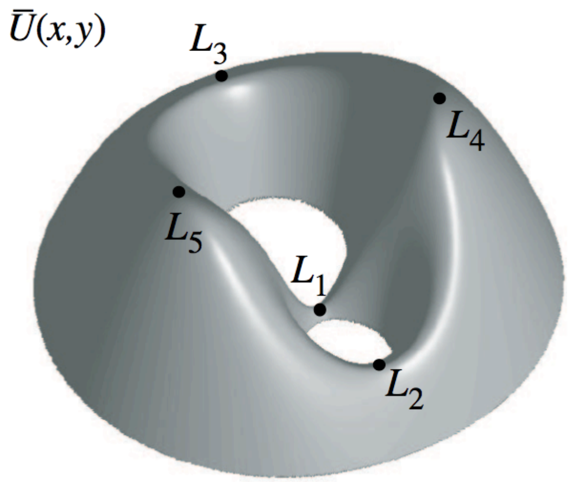


Figure 1: Plot of the effective potential $\overline{U}(x, y)$.

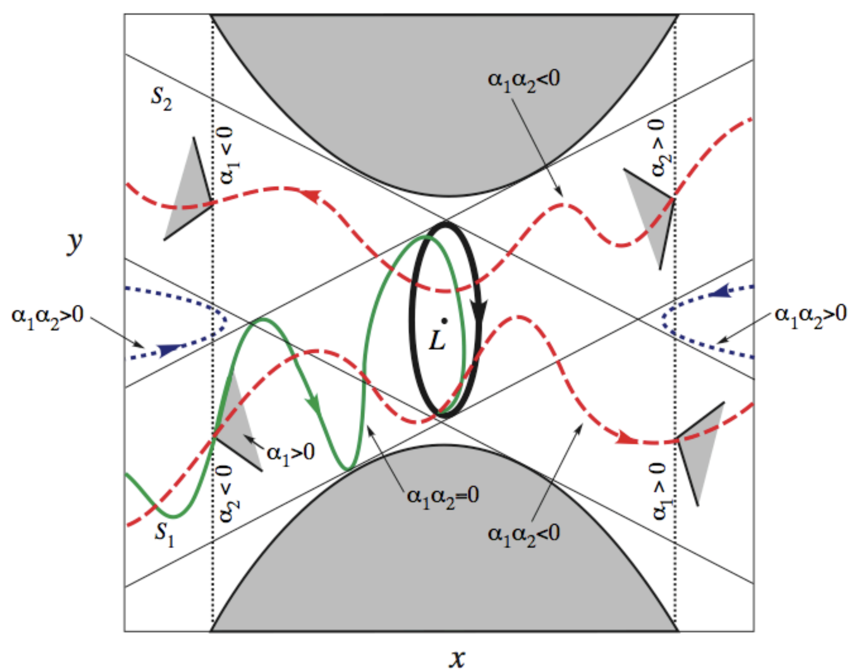


Figure 2: Possible orbits around L_1 or L_2.

It can be shown that four main classes of trajectories around one of the points L_1 or L_2 exist as a function of two parameters depending on the initial conditions (in the graph α_1 and α_2)[1]:

- unstable *periodic* orbits;
- *asymptotic* orbits;
- *transit* orbit;
- *non-transit* orbit;

which obvious meaning in the graph.
Combining the orbits of the two Lagrangian points we ca see how the overall

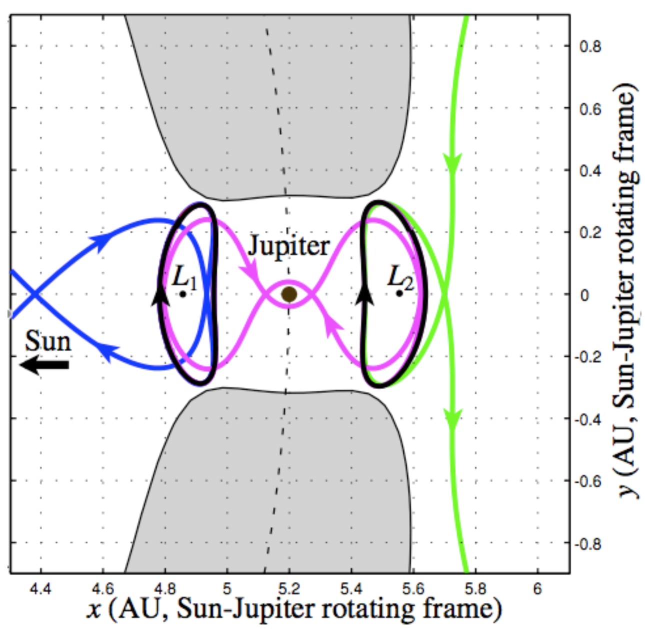


Figure 3: Plot of the semistable orbits.

[1]Of course, being the phase space four-dimensional, the solutions must depend on four parameters and not just two. This is in fact true, but the other two parameters, called in the literature β_1 and β_2 are important just for the early times of the trajectories, while their contribution disappears in the limit $t \to \infty$, which is the limit we are interested in while analyzing classes of trajectories.

pattern looks like.

In the graph the five main domains of the three-body problems (for example,

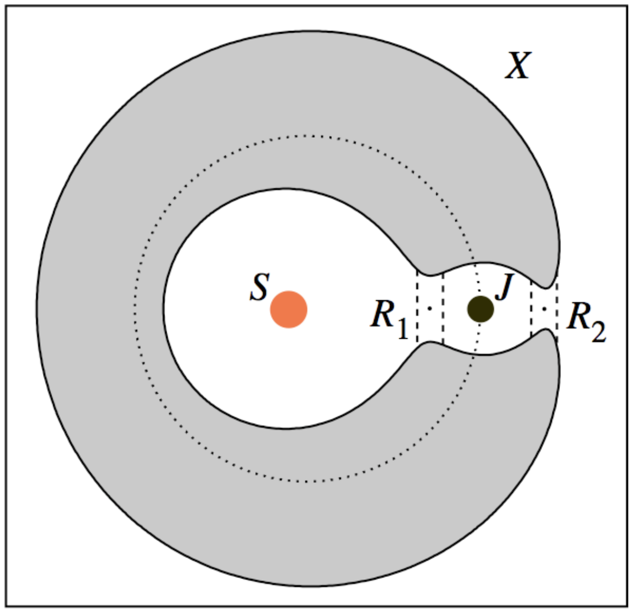
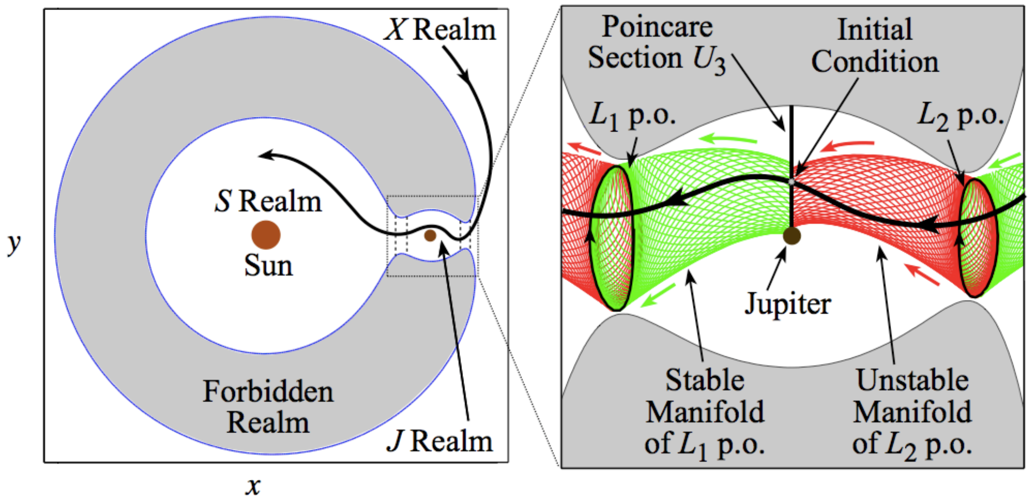


Figure 4: The five gravitational realms.

the Sun-Jupiter system) are shown:

- S is the region where the central body, the Sun, is mostly responsible for the orbit;
- J is the region where Jupiter is seen as the main body;
- X is the region where the orbits mainly see just the center of mass of the Sun-Jupiter system;
- R_1 and R_2 are respectively the region where there are semistable orbits around the Lagrangian points L_1 and L_2.

It is possible to prove that the analyticity of the solutions implies that there are trajectories naturally passing from one domain to another.



(a) (b)

Figure 5: Plot of a trajectory passing from the X realm to the S realm.

As an interesting case, it is important to mention that following a specific trajectory of the unstable manifold[2] of one of the two Lagrangian points it is possible to match precisely a trajectory of the stable manifold of the other Lagrangian point, identifying a path which starts from one unstable asymptotic orbit and ends to a different stable asymptotic orbit.

To better see how the stable and unstable manifolds can be connected and used a short explanation is required. In the figures it is shown a line called Poincaré Section U_3. This line has a particular symmetry and it is useful to be thought as the separation line between the influence of L_1 and L_2 and viceversa. A small change in the momentum on the Poincaré Section can generate a huge change in the final orbit. Also, plotting the phase-space of the orbits for the unstable manifold of L_2 and the stable manifold of L_1 at the Poincaré Section, it is possible to see how for certain orbits it is natural to pass from L_2 to L_1, as the two surfaces in the phase-space overlap.

[2] The (un)stable manifold is the region of the phase-space where the (un)stable asymptotic orbits are located.

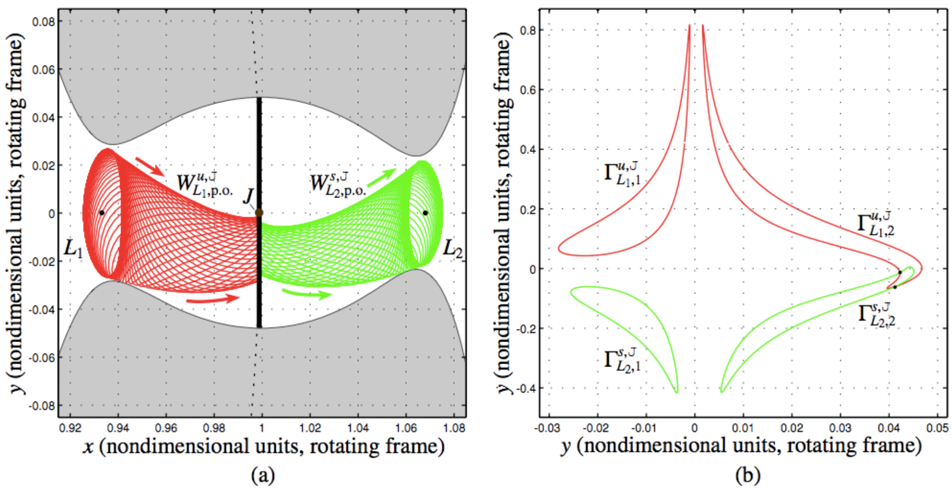


Figure 6: Plot of the phase-space for orbits leaving L_1 and reaching L_2.

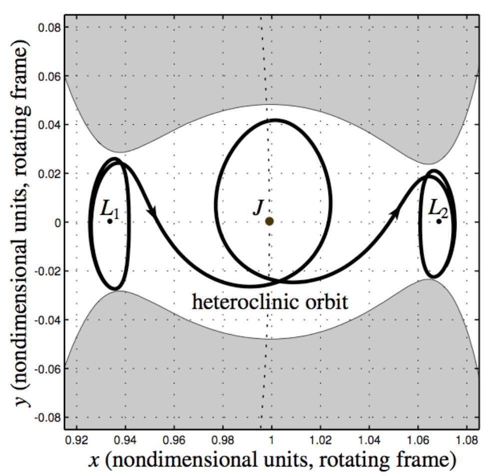


Figure 7: Orbit asymptotically leaving L_1 and asymptotically reaching L_2. At the Poincaré section the position of this orbit in the phase-space is in the region of overlap shown above.

2.4 Chaoticity

The last argument, involving regions overlapping in the phase-space, shows how the system can easily contain chaoticity. Being the equations of motion nonlinear, but well defined and analytical in the entire domain, it is possible to prove how within a region of the phase-space where there are orbits belonging to one of the realms, there can be some "hidden" regions where the orbits eventually change realm. As an example, the plot shows how it

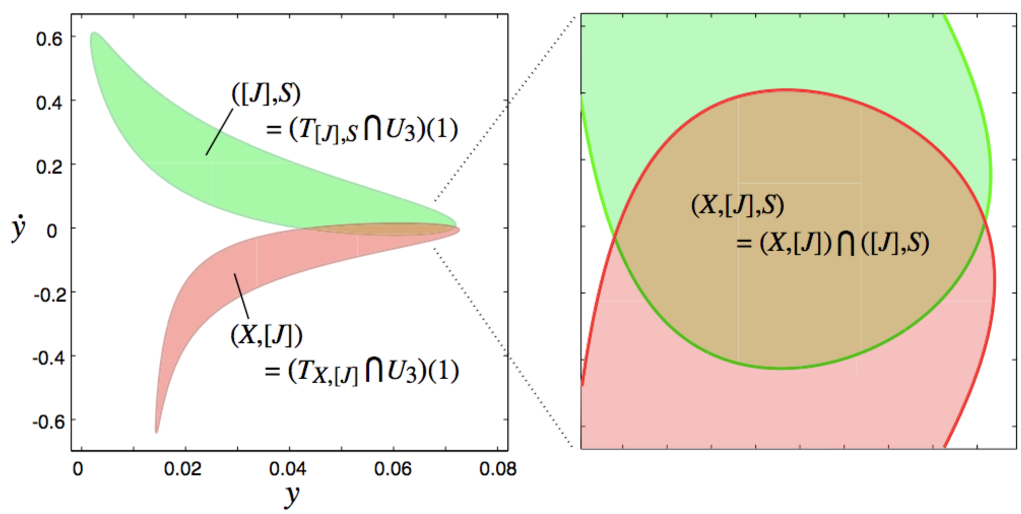


Figure 8: Plot of the region of overlap in the phase-space at the Ponicaré section. The overlap represents the "initial conditions" at the Poincareé section of the trajectory shown in figure 5(a).

is possible to find small regions[3] whose corresponding orbits are extremely unlikely (having random initial conditions), but possible.
Continuing the analysis the system reveals its fractal nature, with orbits that go back to the original realm or change indefinitely realms. The plot below is an example of this feature.

[3]Small here assumes the significance of very precise initial conditions; just a tiny change in them, and the orbit will eventually diverge exponentially.

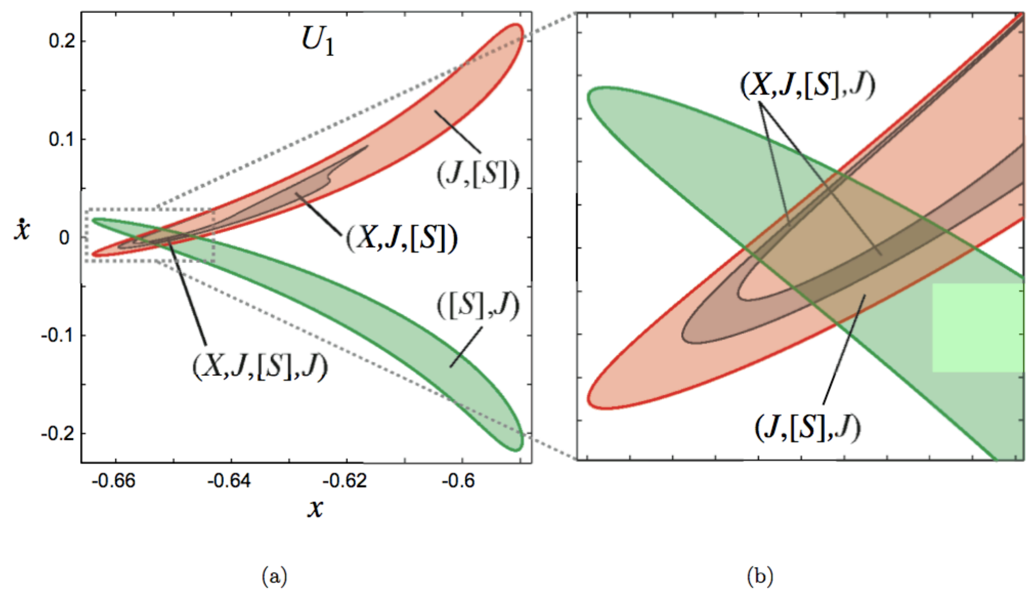


Figure 9: Plot of the fractal nature of the orbits in the phase-space.

3 The four-body problem

The analysis of the planar restricted three-body problem has shown how it is possible to use the Lagrangian points to pass from an "outer" orbit to an "inner" orbit. Combining different three-body problems we can find a way to iterate this method and transferring objects from any to any orbit in the solar system with no or almost no energy cost. A good example is given by the Sun-Earth-Moon system. In the picture (figure 10) it is shown how the three-body problem related to the Earth-Moon system lives actually within the "bigger" three-body problem related to the Sun-Earth system.

The main difference when applying this three-body-problem procedure to the four-body problem is that the total potential becomes time dependent. If the two systems are separate enough, the time dependence influences the procedure of tracking the trajectories only in a synchronization need for the orbits, but the main features remain unchanged. Although, this means that the right alignment must be waited in order to match the different manifolds. Small corrections in the momentum of the orbit can easily increase the window of overlap between the orbits in the phase-space. In the figure it

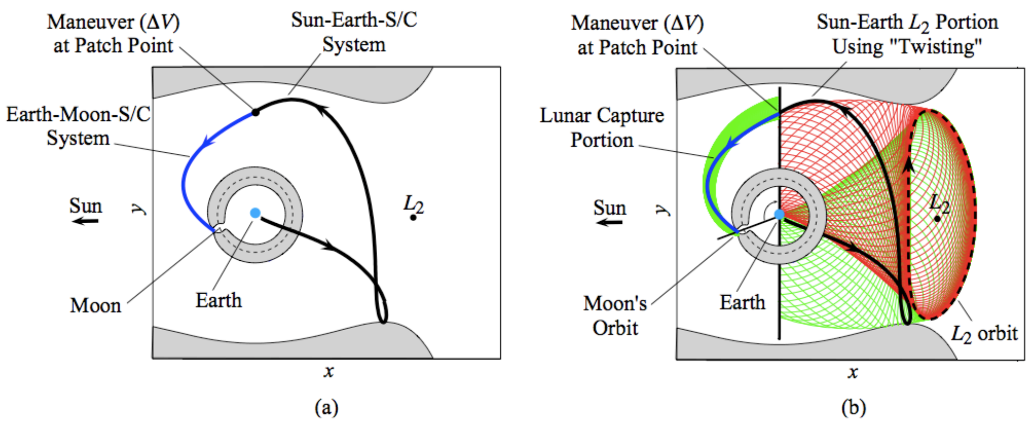


Figure 10: The Sun-Earth-Moon four-body problem seen as two three-body problems.

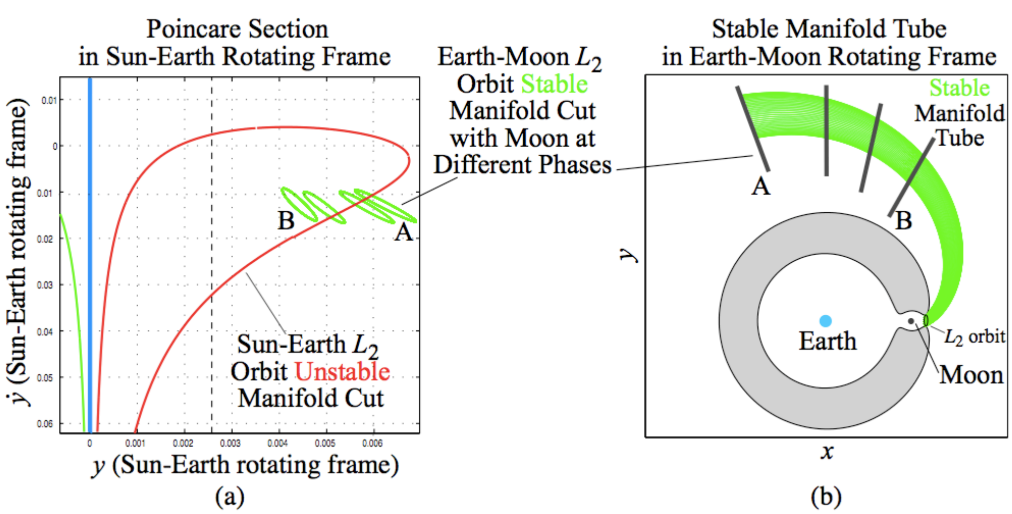


Figure 11: Plot of the time window of the phase-space overlap between the Sun-Earth and the Earth-Moon systems.

is shown how the timing influences the low-fuel (low-change-in-momentum) trajectory of an object leaving the Earth and approaching the Moon.
The Earth-to-Moon trajectory is also shown in the figure to compare the old-

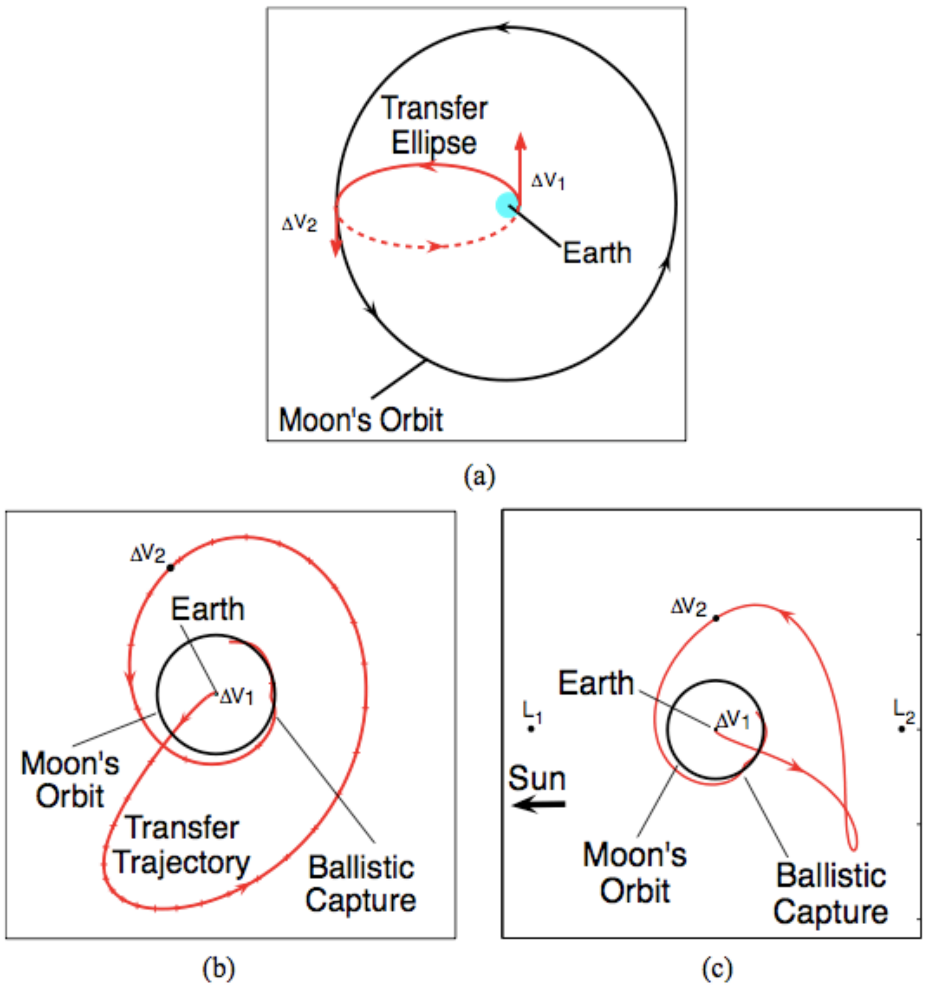


Figure 12: The Earth-to-Moon trajectory.

fashioned approach with the four-body-problem approach. In part *(a)* the standard EtM trajectory, in part *(b)* the four-body-problem trajectories in the Earth-Moon rotating reference frame, in part *(c)* the four-body-problem trajectories in the Sun-Earth rotating reference frame.

4 Results

It has been shown that using the multiple three-body-problem approach it is possible to draw low-fuel trajectories. In the Earth-to-Moon case it is possible to save up to 30% of the fuel required by the old-fashioned trajectory. The 30% is a remarkable saving, especially because the majority of the fuel has still to be used to reach the escape velocity and leave the Earth.
Other very complicated orbits can be calculated with this method, and some of them have been successfully used for probes and space missions.

5 Conclusion

The multiple three-body problem approach allows to draw trajectories from any point of the solar system to any other point with very little change in momentum, implying considerable savings in fuel. The only additional cost is in terms of time. In fact, the nature of the problem makes it time dependent when the bodies involved are more than three and the right alignment might require waiting a long time. Also, some of the trajectories are relatively slow, which can overall makes the time of the completion of the desired trajectories much longer, even if almost free in terms of energy.
The procedure also illustrates how it is possible for small objects as asteroids or comets to change their orbit with respect to the solar system. Historically, the Schumacher-Levi cometh followed one of these complicated paths in the Sun-Jupiter system before being eventually captured by Jupiter, producing the collision we saw a few years ago.
It is possible in the future to put a space station in one of the unstable Lagrangian points with respect to a planet and to use these low-fuel trajectories to send back and forth spaceships to asteroids for collecting minerals with almost no cost in energy. It is also possible to study the possible contamination of life forms from one planet, cometh or asteroid to another, as a possible link for the origin of life.

References

[1] Shane David Ross. *Cylindrical Manifolds and Tube Dynamics in the Restricted Three-Body Problem.* Caltech 2004, PhD thesis.